Math 260 | Feb. 9, 2012

Exam 1

Jerry L. Kazdan
12:00 – 1:20

Directions This exam has 10 questions (10 points each). Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Neatness counts.*

1. Which of the following sets are linear spaces?
   a) The points $X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0$.
   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.
   c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x) \cos 2x \, dx = 0$.
   d) The set of solutions $y = y(t)$ of $y'' + 4y' + y = x^2 - 3$. [Note: You are *not* being asked to solve this differential equation. You are only being asked a more primitive question.]

2. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say V_1 and V_2 are (distinct!) solutions of the equations $LX = Y_1$ while W is a solution of $LX = Y_2$. Answer the following in terms of V_1, V_2, and W.
   a) Find some solution of $LX = 2Y_1 - 3Y_2$.
   b) Find another solution (other than W) of $LX = Y_2$.

3. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.
   a) If $n = k$ there is always *at most one* solution.
   b) If $n > k$, given any Y you can *always* solve $AX = Y$.
   c) If $n > k$ the nullspace of A has dimension *greater* than zero.
   d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.
   e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

4. Find a real 2×2 matrix A such that $A^4 = I$ but $A^2 \neq I$.

5. Find a quadratic polynomial $p(x)$ that passes through the three points $(-1, 0)$, $(0, -1)$, and $(2, 3)$. [Don't bother to "simplify" your answer.]

6. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$ be given matrices, and let $C := BA : \mathbb{R}^3 \to \mathbb{R}^3$. Show that C *cannot* be invertible.

7. In $\mathbb{R}^3$, find the distance from the point $P := (1, 1, 0)$ to the plane $x + 2y - z = 0$.

8. Let U, and V, W be (non-zero) orthogonal vectors and let $Z = aU + bV$, where a and b are scalars.

   a) (Pythagoras) Show that $\|Z\|^2 = a^2\|U\|^2 + b^2\|V\|^2$.

   b) Find a formula for the coefficient a in terms of U and Z only.

9. Let $g(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0, \\ 1 & \text{for } 0 \le x < \pi \end{cases}$, and extend $g(x)$ for all real x so that it is periodic with period 2π. If its Fourier series is $g(x) = \sum_{k=-\infty}^{\infty} c_k \frac{e^{ikx}}{\sqrt{2\pi}}$, find the coefficients c_0 and c_{-2}.

10. A particular solution of $u" + 4u = 2x^2$ is $u_p = \frac{1}{2}x^2 - \frac{1}{4}$. Find a solution that satisfies the initial conditions $u(0) = 0$ and $u'(0) = 0$.