Math 260 — **Exam 1** — Jerry L. Kazdan

Feb. 9, 2012 — 12:00 – 1:20

DIRECTIONS This exam has 10 questions (10 points each). Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Neatness counts.*

1. Which of the following sets are linear spaces?

   a) The points $X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0$.

      SOLUTION: Yes – obviously.

   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.

      SOLUTION: Yes – obviously.

   c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\cos 2x\,dx = 0$.

      SOLUTION: Yes – obviously.

   d) The set of solutions $y = y(t)$ of $y'' + 4y' + y = x^2 - 3$. [NOTE: You are *not* being asked to solve this differential equation. You are only being asked a more primitive question.]

      SOLUTION: No. The function $y(x) \equiv 0$ does not satisfy this equation.

2. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say V_1 and V_2 are (distinct!) solutions of the equations $LX = Y_1$ while W is a solution of $LX = Y_2$. Answer the following in terms of V_1, V_2, and W.

   a) Find some solution of $LX = 2Y_1 - 3Y_2$.

      SOLUTION: For instance $X := 2V_1 - 3W$

   b) Find another solution (other than W) of $LX = Y_2$.

      SOLUTION: For instance $X := V_2 - V_1 + W$.

3. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

      SOLUTION: Counterexample: $A = 0$ matrix.

   b) If $n > k$, given any Y you can *always* solve $AX = Y$.

      SOLUTION: Counterexample: A is the zero matrix.

   c) If $n > k$ the nullspace of A has dimension *greater* than zero.

      SOLUTION: Yes, by the Rank Theorem: $\dim \mathcal{N}(A) = n - \dim \mathcal{I}(A) \ge n - k \ge 1$

   d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

      SOLUTION: Yes, by the Rank Theorem: $k > n = \dim \mathcal{I}(A) + \dim \mathcal{N}(A) \ge \dim \mathcal{I}(A)$. Thus the dimension of the image of A is less than k.

   e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

      SOLUTION: Counterexample: let A be the zero matrix.

4. Find a real 2×2 matrix A such that $A^4 = I$ but $A^2 \neq I$.

SOLUTION: For instance, A is a rotation by $\pi/2$ (90 degrees): $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

5. Find a quadratic polynomial $p(x)$ that passes through the three points $(-1,\,0)$, $(0,\,-1)$, and $(2,\,3)$. [Don't bother to "simplify" your answer.]

SOLUTION: This is simple enough that the computation is fairly easy using many different bases for the space of quadratic polynomials. I'll use a Lagrange basis:

$$p_1(x) := \frac{x(x-2)}{-1(-1-2)}; \quad p_2(x) := \frac{(x+1)(x-2)}{1(0-2)}; \quad p_3(x) := \frac{(x+1)x}{(2+1)2}.$$

Then

$$p(x) = 0p_1(x) + (-1)p_2(x) + 3p_3(x) = \frac{(x+1)(x-2)}{2} + 3\frac{(x+1)x}{6} = x^2 - 1.$$

6. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$ be given matrices, and let $C := BA : \mathbb{R}^3 \to \mathbb{R}^3$. Show that C *cannot* be invertible.

SOLUTION: Any proof that begins with $C^{-1} = A^{-1}B^{-1}$ is almost certainly nonsense since this presumes A and B are invertible. But if a matrix is invertible, it must be square (why?), and neither A nor B are square. Note that $D := AB : \mathbb{R}^2 \to \mathbb{R}^2$ *can* be invertible (Simple Example?).

Here is a solution. If C is invertible, it must be one-to-one. But A can't be one-to-one (why?) and anything in the nullspce of A is also in then nullspace of C.

Another solution. If C is invertible, it must be onto. But B can't be onto (why?) and anything in the image of C must also be in then image of B.

7. In $\mathbb{R}^3$, find the distance from the point $P := (1, 1, 0)$ to the plane $x + 2y - z = 0$.

SOLUTION: There are several approaches. One begins with the observation that the vector $N := (1, 2, -1)$ is orthogonal to this plane because if $V := (x, y, z)$ is in this plane then $\langle V,\, N\rangle = 0$. Think of P as a vector from the origin (which is in the plane) to the point P. From a sketch, it is clear that the distance from P to this plane is $\|P\| \cos\theta$, where θ is the angle between the vectors P and N. But $\langle P,\, N\rangle = \|P\|\|N\| \cos\theta$. Thus

$$\text{Distance} = \frac{\langle P,\, N\rangle}{\|N\|} = \frac{3}{\sqrt{6}}$$

Since we might have used $-N$ for the normal to the plane, to find the distance, if needed we should insert the absolute value in the final step.

8. Let U, and V be (non-zero) orthogonal vectors and let $Z = aU + bV$, where a and b are scalars.

   a) (Pythagoras) Show that $\|Z\|^2 = a^2\|U\|^2 + b^2\|V\|^2$.

   SOLUTION:
$$\begin{aligned}\|Z\|^2 =& \|aU + bV\|^2 = \langle aU + bV,\, aU + bV\rangle \\ =& \langle aU,\, aU\rangle + \langle aU,\, bV\rangle + \langle bV,\, aU\rangle + \langle bV,\, bV\rangle \\ =& a^2\|U\|^2 + b^2\|V\|^2,\end{aligned}$$

   where we used the orthogonality of U and V.

   b) Find a formula for the coefficient a in terms of U and Z only.

   SOLUTION: Take the inner product of both sides of $Z = aU + bV$ with U and use the orthogonality of U and V:
$$\langle Z,\, U\rangle = \langle aU + bV,\, U\rangle = a\|U\|^2 \qquad \text{so} \qquad a = \frac{\langle Z,\, U\rangle}{\|U\|^2}.$$

9. Let $g(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0, \\ 1 & \text{for } 0 \le x < \pi \end{cases}$, and extend $g(x)$ for all real x so that it is periodic with period 2π. If its Fourier series is $g(x) = \sum_{k=-\infty}^{\infty} c_k \frac{e^{ikx}}{\sqrt{2\pi}}$, find the coefficients c_0 and c_{-2}.

SOLUTION: We use the basic formula
$$c_k = \langle g,\, \frac{e^{ikx}}{\sqrt{2\pi}}\rangle = \int_{-\pi}^{\pi} g(x)\frac{e^{-ikx}}{\sqrt{2\pi}}\, dx = \frac{1}{\sqrt{2\pi}}\int_0^{\pi} e^{-ikx}\, dx$$

Thus,
$$c_0 = \frac{\pi}{\sqrt{2\pi}} = \sqrt{\frac{\pi}{2}} \qquad \text{and} \qquad c_{-2} = \frac{1}{\sqrt{2\pi}}\int_0^{\pi} e^{2ix}\, dx = \frac{1}{2i\sqrt{2\pi}}[e^{2i\pi} - 1] = 0.$$

10. A particular solution of $u" + 4u = 2x^2$ is $u_p = \frac{1}{2}x^2 - \frac{1}{4}$. Find a solution that satisfies the initial conditions $u(0) = 0$ and $u'(0) = 0$.

SOLUTION: We first find the general solution, u_h, of the homogeneous equation. Seeking a solution in the form $e^{\lambda x}$ we find $\lambda^2 + 4 = 0$ so $\lambda = \pm 2i$. Thus $u_h(x) = Ae^{2ix} + Be^{-2ix}$ or, in the equivalent real form, $u_h(x) = C\cos 2x + D\sin 2x$, where A, B, C, and D can be any constants. We'll use the real form. The general solution of the inhomogeneous equation is
$$u(x) = C\cos 2x + D\sin 2x + \frac{1}{2}x^2 - \frac{1}{4}.$$

It remains to pick C and D to match the initial conditions:
$$0 = u(0) = C - \frac{1}{4}, \qquad \text{and} \qquad 0 = u'(0) = 2D.$$

Thus $u(x) = \frac{1}{4}\cos 2x + \frac{1}{2}x^2 - \frac{1}{4}$.