Math 260 | **Final Exam** | Jerry L. Kazdan
May 2, 2012 | | 9:00 – 11:00

DIRECTIONS This exam has two parts. PART A has 5 short answer questions (5 points each, so 25 points) while PART B has 8 traditional problems (10 points each, so 80 points). Total: 105 points. *Neatness counts.*

Closed book, no calculators, computers, ipods, cell phones, etc – but you may use one $3'' \times 5''$ card with notes on both sides.

PART A: Five short answer questions (5 points each, so 25 points).

A–1. Let $\mathcal{S}$ be the linear space of 2×2 matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a+d=0$. Compute the dimension of $\mathcal{S}$.

A–2. Let V and W be linear spaces and $L : V \to W$ a linear map. Let w_1 and w_2 be in W. Say $v_1 \in V$ is a solution of $Lv_1 = w_1$ while *both* v_2 and v_3 are *distinct* points in V that satisfy $Lv_2 = Lv_3 = w_2$. Does the equation $Lx = w_1$ have a solution other than v_1? Explain your reasoning.

A–3. Let $f(t)$ be a smooth function of the real variable t. Show that for *any* real constants a and b, the function $u(x,y) := f(ax+by)$ satisfies $u_{xx}u_{yy} - u_{xy}^2 = 0$.

A–4. Consider the surface defined implicitly by $x^2 + 9y^2 - z^2 = 10$. Find a vector orthogonal to the tangent plane at $(1, 1, 0)$.

A–5. Let $J := \displaystyle\int_0^2 \left(\int_0^{x^2} f(x,y)\, dy \right) dx$. Rewrite this as an iterated integral with the order of integration reversed, so one first integrates with respect to x.

PART B: Eight traditional problems (10 points each, so 80 points).

B–1. Consider the set of real-valued continuous functions on the interval $-1 \le x \le 1$ with the inner product $\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\, dx$.

a) Find a quadratic polynomial $p(x) := a + bx + cx^2$ (with $a \ne 0$) that is orthogonal to both $e_1(x) := 1$ and $e_2(x) := x$.

b) Find the orthogonal projection of $q(x) := x^4$ into the subspace spanned by $e_1(x)$, $e_2(x)$, and $p(x)$.

B–2. Find a solution of $u'' + 4u = x^2$ that satisfies the initial conditions $u(0) = 0$ and $u'(0) = 0$.

B–3. Let A be a real $n \times n$ antisymmetric matrix.

a) Show that $\langle X,\, AX\rangle = 0$ for all vectors $X \in \mathbb{R}^n$.

b) Say $X(t)$ is a solution of the differential equation $\dfrac{dX}{dt} = AX$. Show that $\|X(t)\| =$ constant. [REMARK: In the special case $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ this implies $\sin^2 t + \cos^2 t = 1$.]

B–4. Find and classify the critical points of $g(x,y) := x^2 - 2xy + \frac{1}{3}y^3 - 3y$.

B–5. Compute $\displaystyle\oint_\gamma 2x\,dy - y\,dx$ where the closed curve γ is the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$, and $(1,2)$, traversed counterclockwise.

B–6. Let $\mathbf{V} = (y^2 + x)\mathbf{i} + (2xy - 3)\mathbf{j}$.

a) Find a function $u(x,y)$ so that $\mathbf{V} = \nabla u$.

b) Let γ be the triangle bounded by the x-axis, the y-axis, and the straight line $2x + y = 2$, traversed counterclockwise. Compute $\displaystyle\oint_\gamma \mathbf{V}\cdot d\mathbf{s}$.

B–7. Consider the region $\Omega \subset \mathbb{R}^3$ above the surface $z = x^2 + y^2$ and below the plane $z = 4$. Compute $\displaystyle\iiint_\Omega 2z\,dV$.

B–8. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary $\partial\Omega$ and let $w(x,y,t)$ be the solution of the heat equation

$$w_t = \Delta w \quad \text{for all } (x,y) \in \Omega \quad \text{and } t \ge 0, \qquad \text{with } \; w = 0 \quad \text{for } (x,y) \text{ on } \partial\Omega.$$

a) Define $E(t) := \dfrac{1}{2}\displaystyle\iint_\Omega w^2(x,y,t)\,dx\,dy$. Show that $dE/dt \le 0$.

b) If in addition the initial temperature $w(x,y,0) = 0$, show that $w(x,y,t) = 0$ for all $(x,y) \in \Omega$ and $t \ge 0$.

c) If $u(x,y,t)$ and $v(x,y,t)$ both satisfy the heat equation in Ω with $u(x,y,t) = v(x,y,t)$ on $\partial\Omega$ for all $t \ge 0$ and also $u(x,y,0) = v(x,y,0)$, show that $u(x,y,t) = v(x,y,t)$ for all $(x,y) \in \Omega$ and $t \ge 0$.