Math 260 **Final Exam** Jerry L. Kazdan
May 2, 2012 9:00 – 11:00

DIRECTIONS This exam has two parts. PART A has 5 short answer questions (5 points each, so 25 points) while PART B has 8 traditional problems (10 points each, so 80 points). Total: 105 points. *Neatness counts.*

Closed book, no calculators, computers, Pt's, cell phones, etc – but you may use one $3'' \times 5''$ card with notes on both sides.

PART A: Five short answer questions (5 points each, so 25 points).

A–1. Let $\mathcal{S}$ be the linear space of 2×2 matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a+d=0$. Compute the dimension of $\mathcal{S}$.

SOLUTION: Since $d=-a$, then

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so the dimension is 3.

A–2. Let V and W be linear spaces and $L: V \to W$ a linear map. Let w_1 and w_2 be in W. Say $v_1 \in V$ is a solution of $Lv_1 = w_1$ while *both* v_2 and v_3 are *distinct* points in V that satisfy $Lv_2 = Lv_3 = w_2$. Does the equation $Lx = w_1$ have a solution other than v_1? Explain your reasoning.

SOLUTION: Since $L((v_2 - v_3) = 0$, then $L(v_1 + v_2 - v_3) = Lv_1 = w_1$. Thus another solution is $v_1 + (v_2 - v_3)$.

A–3. Let $f(t)$ be a smooth function of the real variable t. Show that for *any* real constants a and b, the function $u(x,y) := f(ax+by)$ satisfies $u_{xx}u_{yy} - u_{xy}^2 = 0$.

SOLUTION: By the chain rule:

$$u_x(x,y) = f'(ax+by)a, \qquad u_y(x,y) = f'(ax+by)b,$$
$$u_{xx}(x,y) = f''(ax+by)a^2, \quad u_{xy}(x,y) = f''(ax+by)ab, \quad \text{and} \quad u_{yy}(x,y) = f''(ax+by)b^2.$$

The result is now clear.

A–4. Consider the surface defined implicitly by $x^2 + 9y^2 - z^2 = 10$. Find a vector orthogonal to the tangent plane at $(1, 1, 0)$.

SOLUTION: If a surface is defined implicitly by $f(x,y,z) = c$, so it is a level surface of f, then its gradient, ∇f, is perpendicular to the surface (that is, it is orthogonal to the tangent plane). Since $\nabla(x^2 + 9y^2 - z^2) = (2x,\, 18y, -2z)$, then at the given point $N := (2,\, 18, 0)$ is orthogonal to the tangent plane.

A–5. Let $J := \int_0^2 \left(\int_0^{x^2} f(x,y)\,dy \right) dx$. Rewrite this as an iterated integral with the order of integration reversed, so one first integrates with respect to x.

SOLUTION: The region of integration is bounded on the bottom by $y = 0$, on the left by the curve $y = x^2$, and on the right by $x = 2$. This if we interchange the order of integration

$$J = \int_0^4 \left(\int_{\sqrt{y}}^2 f(x,y)\,dx \right) dy.$$

PART B: Eight traditional problems (10 points each, so 80 points).

B–1. Consider the set of real-valued continuous functions on the interval $-1 \le x \le 1$ with the inner product $\langle f,\, g\rangle := \int_{-1}^1 f(x)g(x)\,dx$.

a) Find a quadratic polynomial $p(x) := a + bx + cx^2$ (with $a \neq 0$) that is orthogonal to both $e_1(x) := 1$ and $e_2(x) := x$.

SOLUTION: We want $\langle p,\, e_1\rangle = 0$ and $\langle p,\, e_2\rangle = 0$. But

$$\langle p,\, e_1\rangle = \int_{-1}^1 (a + bx + cx^2)\,dx = 2(a + \tfrac{1}{3}c)$$
$$\langle p,\, e_2\rangle = \int_{-1}^1 (a + bx + cx^2)x\,dx = \tfrac{2}{3}b.$$

Thus $c = -3a$ and $b = 0$. For instance, $p(x) = 1 - 3x^2$

b) Find the orthogonal projection of $q(x) := x^4$ into the subspace $\mathcal{S}$ spanned by $e_1(x)$, $e_2(x)$, and $p(x)$.

SOLUTION: We want to write $x^4 = Ae_1 + Be_2 + Cp + w$ where w is orthogonal to e_1, e_2, and p. Thus

$$A = \frac{\langle x^4,\, e_1\rangle}{\|e_1\|^2}, \qquad B = \frac{\langle x^4,\, e_2\rangle}{\|e_2\|^2}, \qquad C = \frac{\langle x^4,\, p\rangle}{\|p\|^2}.$$

The computation is now straightforward – but tedious:

$$\|e_1\|^2 = \int_{-1}^1 1^2\,dx = 2, \qquad \|e_2\|^2 = \int_{-1}^1 x^2\,dx = \frac{2}{3}$$
$$\|p\|^2 = \int_{-1}^1 (1 - 3x^2)^2\,dx = \int_{-1}^1 (1 - 6x^2 + 9x^4)\,dx = \frac{8}{5}$$
$$\langle x^4,\, e_1\rangle = \int_{-1}^1 x^4\,dx = \frac{2}{5}, \qquad \langle x^4,\, e_2\rangle = \int_{-1}^1 x^5\,dx = 0$$
$$\langle x^4,\, p\rangle = \int_{-1}^1 x^4 - 3x^6\,dx = \frac{2}{5} - \frac{6}{7} = -\frac{16}{35}.$$

Consequently

$$A = \frac{1}{5}, \qquad B = 0, \qquad C = -\frac{2}{7}$$

so

$$\text{Projection}_{\mathcal{S}}(x^4) = \frac{1}{5} - \frac{2}{7}(1 - 3x^2) = -\frac{3}{35} + \frac{6}{7}x^2.$$

B–2. Find a solution of $u'' + 4u = x^2$ that satisfies the initial conditions $u(0) = 0$ and $u'(0) = 0$.

SOLUTION: Seek a particular solution, u_p of the inhomogeneous equation in the form $u_p(x) = a + bx + cx^2$. Substituting this into the equation we get

$$2c + 4(a + bx + cx^2) = x^2.$$

Thus $c = 1/4$ so $a = -1/8$ and $b = 0$. This gives $u_p(x) = -\frac{1}{8} + \frac{1}{4}x^2$.

Since the general solution of the homogeneous equation is $u_h(x) = A\cos 2x + B\sin 2x$, the general solution of the inhomogeneous equation is

$$u(x) = -\frac{1}{8} + \frac{1}{4}x^2 + A\cos 2x + B\sin 2x.$$

We pick the constants A and B to match the initial conditions:

$$0 = u(0) = -\frac{1}{8} + A, \quad \text{and} \quad 0 = u'(0) = 2B$$

which gives $\boxed{u(x) = -\frac{1}{8} + \frac{1}{4}x^2 + \frac{1}{8}\cos 2x}$.

B–3. Let A be a real $n \times n$ antisymmetric matrix.

a) Show that $\langle X,\, AX\rangle = 0$ for all vectors $X \in \mathbb{R}^n$.

SOLUTION: By definition of A^* and symmetry of the inner product we have $\langle X,\, AX\rangle = \langle A^*X,\, X\rangle = -\langle AX,\, X\rangle = -\langle X,\, AX\rangle$. Thus $2\langle X,\, AX\rangle = 0$.

b) Say $X(t)$ is a solution of the differential equation $\dfrac{dX}{dt} = AX$. Show that $\|X(t)\| =$ constant. [REMARK: In the special case $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ this implies $\sin^2 t + \cos^2 t = 1$.]

SOLUTION: By part a),

$$\frac{d\|X(t)\|^2}{dt} = 2\langle X(t),\, X'(t)\rangle = 2\langle X(t),\, AX(t)\rangle = 0.$$

B–4. Find and classify the critical points of $g(x, y) := x^2 - 2xy + \frac{1}{3}y^3 - 3y$.

SOLUTION: At a critical point: $0 = g_x = 2x - 2y$ and $0 = g_y = -2x + y^2 - 3$. The first equation gives $x = y$. Using this in the second equation we find $0 = y^2 - 2y - 3 = (y-3)(y+1)$. Consequently there are two critical points: $(3,\, 3)$ and $(-1,\, -1)$.

To classify these we compute the second derivative (Hessian) matrix

$$g''(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 2y \end{pmatrix}.$$

This gives

$$g''(3,3) = \begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix} \quad \text{and} \quad g''(-1,\,-1) = \begin{pmatrix} 2 & -2 \\ -2 & -2 \end{pmatrix}.$$

The first matrix is positive definite so $(3,\,3)$ is a local minimum. The second is indefinite (and non-degenerate) so $(-1,\,-1)$ is a saddle point.

B–5. Compute $\displaystyle\oint_\gamma 2x\,dy - y\,dx$ where the closed curve γ is the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$, and $(1,2)$, traversed counterclockwise.

Solution: The simplest approach is to use Stokes' Theorem, letting Ω be the interior of the triangle. This gives

$$\oint_\gamma 2x\,dy - y\,dx = \iint_\Omega (2+1)\,dA = 3\text{Area}\,(\Omega) = 3.$$

B–6. Let $\mathbf{V} = (y^2+x)\mathbf{i} + (2xy-3)\mathbf{j}$.

a) Find a function $u(x,y)$ so that $\mathbf{V} = \nabla u$.

Solution: We want $u_x = y^2 + x$ and $u_y = 2xy - 3$. Integrating the first of these with respect to x we obtain $u(x,y) = xy^2 + \frac{1}{2}x^2 + h(y)$. Substituting this into the second equation we find $2xy + h'(y) = 2xy - 3$. Thus $h'(y) = -3$ so $h(y) = -3y + C$, where C is any constant. Consequently

$$u(x,y) = xy^2 + \frac{1}{2}x^2 - 3y + C.$$

It is straightforward to check that this works. [In finding potential functions one almost always ignores the constant C].

b) Let γ be the triangle bounded by the x-axis, the y-axis, and the straight line $2x+y=2$, traversed counterclockwise. Compute $\displaystyle\oint_\gamma \mathbf{V}\cdot d\mathbf{s}$.

Solution: First note that if $\gamma(t)$, $a \le t \le b$is any smooth curve, not necessarily closed, then

$$\int_\gamma \nabla u \cdot d\mathbf{s} = u(\gamma(b)) - u(\gamma(a)).$$

In the special case of a closed curve, $\gamma(b) = \gamma(a)$ so the integral is zero.

This is also an immediate consequence of Stokes' Theorem.

B–7. Consider the region $\Omega \subset \mathbb{R}^3$ above the surface $z = x^2 + y^2$ and below the plane $z = 4$. Compute $J := \iiint_\Omega 2z\, dV$.

SOLUTION: There are many ways to do this. We use cylindrical coordinates. The paraboloid $z = x^2 + y^2$ intersects the plane $z = 4$ in the circle $x^2 + y^2 = 4$, that is, $r = 2$. Thus

$$J = \int_0^{2\pi} \left[\int_0^2 \left(\int_{r^2}^4 2z\, dz \right) r dr \right] d\theta.$$

Now

$$\int_{r^2}^4 2z\, dz = z^2 \Big|_{r^2}^4 = 16 - r^4.$$

Then

$$\int_0^2 (16 - r^4) r\, dr = 8r^2 - \frac{r^6}{6} \Big|_0^2 = 32 - \frac{32}{3} = \frac{64}{3}.$$

Finally

$$J = \frac{64}{3} \int_0^{2\pi} d\theta = \frac{128\pi}{3}.$$

B–8. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary $\partial\Omega$ and let $w(x, y, t)$ be the solution of the heat equation

$$w_t = \Delta w \quad \text{for all } (x, y) \in \Omega \quad \text{and } t \geq 0, \qquad \text{with } w = 0 \quad \text{for } (x, y) \text{ on } \partial\Omega.$$

a) Define $E(t) := \frac{1}{2} \iint_\Omega w^2(x, y, t)\, dx\, dy$. Show that $dE/dt \leq 0$.

SOLUTION: Using the heat equation and Green's first identity

$$\begin{aligned} \frac{dE}{dt} &= \iint_\Omega u(x, y, t) u_t(x, y, t)\, dA \\ &= \iint_\Omega u \Delta u\, dA \\ &= \int_{\partial\Omega} u \frac{\partial u}{\partial n}\, ds - \iint_\Omega |\nabla u|^2\, dA. \end{aligned}$$

Since $u = 0$ on $\partial\Omega$, the first integral on the last line is zero. Because the second term is not positive we conclude that $dE/dt \leq 0$.

b) If in addition the initial temperature $w(x, y, 0) = 0$, show that $w(x, y, t) = 0$ for all $(x, y) \in \Omega$ and $t \geq 0$.

SOLUTION: From its definition, $E(t) \geq 0$. Because $w(x, y, 0) = 0$ we see that $E(0) = 0$. But $dE/dt \leq 0$ so $E(t) = 0$ for all $t \geq 0$. In turn this implies that $w(x, y, t) = 0$ for all $t \geq 0$ since otherwise $E(t)$ would be positive for some $t \geq 0$.

c) If $u(x, y, t)$ and $v(x, y, t)$ both satisfy the heat equation in Ω with $u(x, y, t) = v(x, y, t)$ on $\partial\Omega$ for all $t \geq 0$ and also $u(x, y, 0) = v(x, y, 0)$, show that $u(x, y, t) = v(x, y, t)$ for all $(x, y) \in \Omega$ and $t \geq 0$.

SOLUTION: Let $w(x, y, t) := u(x, y, t) - v(x, y, t)$ and apply the result of part b).