Math 260, Spring 2012 Jerry L. Kazdan

Problem Set 10

DUE: Thurs. March 29. *Late papers will be accepted until 1:00 PM Friday.*

Unless otherwise stated use the standard Euclidean *norm.*

1. [Marsden-Tromba, p. 282 #6] Sketch the solid whose volume is given by

$$\int_0^3 \left(\int_0^2 (9 + x^2 + y^2)\, dx \right)\, dy.$$

2. [Marsden-Tromba, p. 282 #12] Let $f(x, y)$ be continuous on the rectangle $[a, b] \times [c, d]$. For $a < x < b$, $c < y < d$ define

$$G(x, y) := \int_a^x \left(\int_c^y f(u, v)\, dv \right)\, du.$$

Show that $\partial^2 G/\partial x \partial y = \partial^2 G/\partial y \partial x = f(x, y)$.

3. [Marsden-Tromba, p. 289 #9] Let D be the region bounded by the y axis and the parabola $x = -4y^2 + 3$. Compute

$$\iint_D x^3 y\, dx\, dy.$$

4. [Marsden-Tromba, p. 293 #5] Change the order of integration and evaluate $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy$.

5. [Marsden-Tromba, p. 304 #25] Let $W = (x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} \le z \le 1$. Write the following triple integral as an iterated integral:

$$\iiint_W f\, dV = \int_a^b \left\{ \int_?^? \left(\int_?^? f(x, y, z)\, dz \right)\, dy \right\} dx.$$

6. [Marsden-Tromba, p. 304 #30] Let W be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$. Evaluate $\iiint_W x\, dx\, dy\, dz$.

7. a) If A is an $n \times n$ matrix and $c \in \mathbb{R}$, show that $\det(cA) = c^n \det A$.

   b) Let A and B be $n \times n$ matrices. We say that they are *similar* if there is an invertible matrix S such that $A = SBS^{-1}$. If A and B are similar, show that $\det A = \det B$.

8. Compute the determinant of: $A := \begin{pmatrix} a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & -b \\ c & 0 & 0 & 1 & -a \\ d & e & 1 & f & g \end{pmatrix}$.

9. A square matrix A is *upper triangular* if all the elements below the main diagonal are zero,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdot \\ 0 & 0 & \ddots & \cdot \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

If A is upper triangular, show that its determinant is simply the product of the diagonal elements:

$$\det A = a_{11}a_{22}\ldots a_{nn}.$$

10. [Marsden-Tromba, p. 327 #19] Calculate $\iint_\Omega (x+y)^2 e^{x-y}\, dx\, dy$, where Ω is the region bounded by $x+y=1$, $x+y=4$, $x-y=-1$, and $x-y=1$.

11. [Marsden-Tromba, p. 327 #21] Integrate $x^2+y^2+z^2$ 0ver the cylinder $x^2+y^2 \le 2$, $-2 \le z \le 3$.

12. a) Compute $\iint_{\mathbb{R}^2} \frac{dx\, dy}{[1+x^2+y^2]^2}$.

    b) Compute $\iint_{\mathbb{R}^2} \frac{dx\, dy}{[1+x^2+4y^2]^2}$.

    c) Compute $\iint_{\mathbb{R}^2} \frac{dx\, dy}{[1+x^2+2xy+5y^2]^2}$. HINT: $x^2-2xy+5y^2 = (x-y)^2 + \,?$.

    d) Let A be a positive definite 2×2 symmetric real matrix and $X := (x_1,\, x_2) \in \mathbb{R}^2$. Show that

$$\iint_{\mathbb{R}^2} \frac{dx_1\, dx_2}{[1+\langle X,\, AX\rangle]^2} = \frac{\pi}{\sqrt{\det A}}.$$

13. Compute $\iint_{\mathbb{R}^2} e^{-(ax^2+by^2)}\, dx\, dy$, where $a>0$ and $b>0$.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 a) Let A be a real symmetric positive definite 2×2 matrix. Using the method of Problem 12c-d (completing the square), show that

$$\iint_{\mathbb{R}^2} e^{-\langle X,\, AX\rangle}\, dx_1\, dx_2 = \frac{\pi}{\sqrt{\det A}}.$$

b) **Important Fact:** If A is a symmetric matrix, then there is an orthogonal matrix R and a diagonal matrix D so that $A = RDR^{-1}$. In particular, every symmetric matrix is similar to some diagonal matrix.

Use this to show that if A is a positive definite symmetric $n \times n$ matrix, then

$$\int \cdots \int_{\mathbb{R}^n} e^{-\langle X,\, AX\rangle}\, dx_1 \cdots dx_n = \frac{\pi^{n/2}}{\sqrt{\det A}}.$$

[Last revised: March 26, 2012]