Problem Set 3

DUE: In class Thursday, Feb. 2. *Late papers will be accepted until 1:00 PM Friday.*

1. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

   b) If $n > k$ you can *always* solve $AX = Y$.

   c) If $n > k$ the nullspace of A has dimension greater than zero.

   d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

   e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

2. Let A and B be $n \times n$ matrices with $AB = 0$. Give a proof or counterexample for each of the following.

   a) $BA = 0$

   b) Either $A = 0$ or $B = 0$ (or both).

   c) If B is invertible then $A = 0$.

   d) There is a vector $V \neq 0$ such that $BAV = 0$.

3. Consider the system of equations

$$\begin{aligned} x + y - \ z &= a \\ x - y + 2z &= b. \end{aligned}$$

   a) Find the general solution of the homogeneous equation.

   b) A particular solution of the inhomogeneous equations when $a = 1$ and $b = 2$ is $x = 1, y = 1, z = 1$. Find the most general solution of the inhomogeneous equations.

   c) Find some particular solution of the inhomogeneous equations when $a = -1$ and $b = -2$.

   d) Find some particular solution of the inhomogeneous equations when $a = 3$ and $b = 6$.

   [Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

4. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

   a) Find the general solution $\mathbf{Z}$ of the homogeneous equation $A\mathbf{Z} = 0$.

b) Find some solution of $A\mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c) Find the general solution of the equation in part b).

d) Find some solution of $A\mathbf{X} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and of $A\mathbf{X} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

e) Find some solution of $A\mathbf{X} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

f) Find some solution of $A\mathbf{X} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. [Note: $\left(\begin{smallmatrix} 7 \\ 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) + 2\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right)$].

[Remark: After you have done parts a), b) and e), it is possible immediately to write the solutions to the remaining parts.]

5. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$ and $AB : \mathbb{R}^2 \to \mathbb{R}^2$.

   a) Show that BA can *not* be invertible.

   b) Give an example showing that AB might be invertible.

6. Given the five data points:

$$P_1 = (-1, 1), \quad P_2 = (0, 0), \quad P_3 = (1, 0), \quad P_4 = (2, 2), \quad P_5 = (3, 0),$$

find the (unique!) quartic polynomial $p(x)$ that passes through these points. [Don't bother to "simplify" your answer.]

The next sequence of problems involve techniques for explicitely solving the ordinary differential equation

$$Lu := a(x)u'' + b(x)u' + c(x)u = f(x)$$

in the very special (but important) *special case where the coefficients* $a(x)$, $b(x)$, *and* $c(x)$ *are* constants *with* $a \neq 0$, *and the right hand side* $f(x)$ *is simple. These assume you have mastered the ideas concerning the complex exponential* e^{x+iy} *in*
`http://www.math.upenn.edu/ kazdan/260S12/hw/hw0.pdf`

7. Homogeneous equation example

   a) Let $Lu := u'' + u' - 2u$. Find two linearly independent solutions of the homogeneous equation of the form $u(x) = e^{rx}$ where r is a constant, possibly complex.

   b) Seek (and find) solutions u(t) of $u'' + 2u' + 5u = 0$ in the form $u(t) = e^{rx}$, where r might be a complex number. Use this to find two linearly independent real solutions. [See Homework Set 0].

   c) Find a solution of $u'' + 2u' + 5u = 0$ that satisfies the initial conditions $u(0) = 1$, $u'(0) = 0$.

8. HOMOGENEOUS EQUATION Let $Lu := au'' + bu' + cu$ where the coefficients are real constants with $a \neq 0$. Show that $L(e^{rx}) = p(r)e^{rx}$, where $p(r)$ is a quadratic polynomial. Assume the roots of $p(r) = 0$ are distinct

   a) Find two linearly independent solutions (possibly complex) of the homogeneous equation $Lu = 0$.

   b) If the solutions you just found are complex-valued functions, use them to find two linearly independent real solutions.

9. INHOMOGENEOUS EQUATION EXAMPLE: Find some particular solution $v(x)$ of $v'' + v = x^2 - 1$. [SUGGESTION: Since the right hand side is a quadratic polynimial and the coefficients of the differential equation are constants, seek $v(x)$ as a quadratic polynomial: $v(x) = A + Bx + Cx^2$].

   This experiment leads one to the general approach of the next problem on finding a particular solution of the inhomogeneous equation when $f(x)$ is a polynomial.

10. INHOMOGENEOUS EQUATION: POLYNOMIAL Let $\mathcal{P}_N$ be the linear space of polynomials of degree at most N and $L : \mathcal{P}_N \to \mathcal{P}_N$ the linear map defined by $Lu := au'' + bu' + cu$, where a, b, and c are constants. Assume $c \neq 0$ (and $a \neq 0$).

    a) Compute $L(x^k)$.

    b) Show that nullspace (=kernel) of $L : \mathcal{P}_N \to \mathcal{P}_N$ is 0. [A strict proof uses induction – but it is convincing enough to treat the case $N = 3$.]

    c) Show that for every polynomial $q(x) \in \mathcal{P}_N$ there is one and only one solution $p(x) \in \mathcal{P}_N$ of the ODE $Lp = q$. [A strict proof uses induction – but it is convincing enough to treat the case $N = 3$.]

11. INHOMOGENEOUS EQUATION EXAMPLE: Use the observation in Problem 8 to find particular solutions of

    a) $u'' - 4u = 2e^{3x}$

    b) $u'' - 4u = \cos x$ [HINT: $\cos x$ is the real part of e^{ix}.]

    c) $u'' - 4u = \cos x + 2\sin x$

    d) $u'' - 4u = e^{x}\cos x$. [Not assigned – but useful.]

12. Let $u_p(t)$ be a particular solution of the inhomogeneous equation $Lu := u'' + bu' + cu = f(t)$, where b and c are real constants. Assuming $u_p(t)$ is bounded for all $t \geq 0$ (that is, for some constant M we have $|u_p(t)| \leq M$ for all $t \geq 0$), find the conditions on the coefficients b and c that guarentee that *all* solutions of $Lu = f$ are bounded for all $t \geq 0$.

Bonus Problems

[Please give these directly to Professor Kazdan]

1-B [Error in Interpolation] Let $x_0 < x_1 < x_2$ be distinct real numbers and $f(x)$ a smooth function. In class we showed there is a unique quadratic polynomial $p(x)$ with the property that $p(x_j) = f(x_j)$ for $j = 0, 1, 2$. Here you find a formula for the error:$= f(x) - p(x)$.

If b is in the open interval (x_0, x_2) with $b \neq x_j$, $j = 0, 1, 2$, show there is a point c (depending on b) in the interval (x_0, x_2) so that

$$f(b) = p(b) + \frac{f'''(c)}{3!}(b - x_0)(b - x_1)(b - x_2).$$

This estimate is related to the procedure used to find the remainder in a Taylor polynomial.

[Suggestion: Define the constant M by

$$f(b) = p(b) + M(b - x_0)(b - x_1)(b - x_2),$$

and look at

$$g(x) := f(x) - [p(x) + M(x - x_0)(x - x_1)(x - x_2)].$$

Now observe that $g(x) = 0$ at x_0, x_1, x_2, and b (by definition of M).]

2-B Let $Lu := u'' + bu' + cu = 0$, where b and c are constants.

a) If $w(x)$ is a solution of the homogeneous equation $Lw = 0$ with initial conditions $w(0) = 0$ and $w'(0) = 0$, show that $w(x) = 0$ for all $x \geq 0$.

b) Make the change of variable $x = -t$and show that as a function of t w satisfies:

$$\frac{d^2w}{dt} - b\frac{dw}{dt} + cw = 0 \quad \text{with} \quad w(0) = 0 \quad \text{and} \quad w'(0) = 0.$$

This has the same structure as the original equation, only the sign of b is flipped so by applying part a), conclude that $w(x) = 0$ for all x.

c) Use this to state and prove a uniqueness theorem for the inhomoheneous equation $Lu = f(x)$ with $u(0) = \alpha$ and $u'(0) = \beta$

[Last revised: May 23, 2012]