Problem Set 4

DUE: Never [Exam 1 is on Thursday, Feb.9, 12:00-1:20]

Unless otherwise stated use the standard Euclidean *norm.*

1. In $\mathbb{R}^2$, define the new norm of a vector $V = (x, y)$ by $\|V\| := 2|x| + |y|$. Show this satisfies the properties of a norm. [REMARK: This is a "taxicab" norm where, because of traffic it is twice as expensive to go across town than up town.]

2. a) In $\mathbb{R}^3$, find the distance from the point $(1, 1, 1)$ to the plane $x + 2y - z = 3$.
   b) In $\mathbb{R}^4$, compute the distance from the point $(1, -2, 0, 3)$ to the hyperplane $x_1 + 3x_2 - x_3 + x_4 = 3$.

3. a) In $\mathbb{R}^3$, find an orthogonal basis for the plane $x + 2y - z = 0$.
   b) Use it to find the orthogonal projection of $V = (1, 0, -1)$ into this plane.
   c) What is the orthogonal projection of V perpendicular to this plane?

4. In $\mathbb{R}^5$,write $X = (x_1, x_2, x_3, x_4, x_5)$ and let S be the subspace spanned by the vectors

$$V_1 := (1, 0, 0, 0, 0), \quad V_2 := (0, 3, 4, 0, 0), \quad V_3 := (0, 4, -3, 0, 0).$$

   Find the orthogonal projection $P_S(X)$ of X into S and the projection of X orthogonal to S.

5. Let V, W be vectors in $\mathbb{R}^n$ with the usual Euclidean norm. Prove the *parallelogram identity* $\|V + W\|^2 + \|V - W\|^2 = 2\|V\|^2 + 2\|W\|^2$ and interpret it geometrically.

6. Find all vectors in the plane (through the origin) spanned by $\mathbf{V} = (1, 1 - 2)$ and $\mathbf{W} = (-1, 1, 1)$ that are perpendicular to the vector $\mathbf{Z} = (2, 1, 2)$.

7. In $\mathbb{R}^3$, let V be a non-zero vector and X_0 and P points.
   a) Find the equation of the plane through X_0 that is orthogonal to V, so V is a *normal vector* to this plane.
   b) Compute the distance from this plane to the origin and also to the point P.
   c) Let S be the sphere centered at P with radius r. For which value(s) of r is this sphere tangent to the above plane?

8. Let U, V, W be orthogonal vectors and let $Z = aU + bV + cW$, where a, b, c are scalars.

a) (Pythagoras) Show that $\|Z\|^2 = a^2\|U\|^2 + b^2\|V\|^2 + c^2\|W\|^2$.

b) Find a formula for the coefficient a in terms of U and Z only. Then find similar formulas for b and c. [Suggestion: take the inner product of $Z = aU + bV + cW$ with U].

REMARK The resulting simple formulas are one reason that orthogonal vectors are easier to use than more general vectors. This is vital for Fourier series.

c) Solve the following equations:

$$\begin{aligned} x+y+z+w &= 2 \\ x+y-z-w &= 3 \\ x-y+z-w &= 0 \\ x-y-z+w &= -5 \end{aligned}$$

[Suggestion: Observe that the columns vectors in the coefficient matrix are orthogonal.]

9. Find the (orthogonal) projection of $\mathbf{x} := (1,2,0)$ into the following subspaces:
   a) The line spanned by $\mathbf{u} := (1,1,-1)$.
   b) The plane spanned by $\mathbf{u} := (0,1,0)$ and $\mathbf{v} := (0,0,-2)$
   c) The plane spanned by $\mathbf{u} := (0,1,1)$ and $\mathbf{v} := (0,1,-2)$
   d) The plane spanned by $\mathbf{u} := (1,0,1)$ and $\mathbf{v} := (1,1,-1)$
   e) The plane spanned by $\mathbf{u} := (1,0,1)$ and $\mathbf{v} := (2,1,0)$.
   f) The subspace spanned by $\mathbf{u} := (1,0,1)$, $\mathbf{v} := (2,1,0)$ and $\mathbf{w} := (1,1,0)$.

10. Using the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$, for which values of the real constants α, β, γ are the quadratic polynomials $p_1(x) = 1$, $p_2(x) = \alpha + x$ $p_3(x) = \beta + \gamma x + x^2$ orthogonal? [ANSWER: $p_2(x) = x$, $p_3(x) = x^2 - 1/3$.]

11. Find the function $f \in \operatorname{span}\{1 \sin x, \cos x\}$ that minimizes $\|\sin 2x - f(x)\|$, where the norm comes from the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx \quad \text{on} \quad C[-\pi,\pi].$$

12. Find particular solutions of

$$a).\ \ u'' - u = \cos 2x, \quad b).\ \ u'' - u = \sin 2x, \quad c).\ \ u'' - u = 2\cos 2x - 5\sin 2x.$$

13. A particular solution of $u" + 4u = 2x^2$ is $u_p = \frac{1}{2}x^2 - 1$. Find a solution that satisfies the initial conditions $u(0) = 1$ and $u'(0) = 2$.

14. Find a solution of $u'' - 2u' + 2u = \cos 3x$ that satisfies the initial conditions $u(0) = 0$ and $u'(0) = 0$.

15. a) Let $f(x) = \begin{cases} -1 & \text{for } -\pi \le x < 0, \\ 1 & \text{for } 0 \le x < \pi \end{cases}$. Find its Fourier series (either using trig functions or the complex exponential).

b) Let $g(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0, \\ 1 & \text{for } 0 \le x < \pi \end{cases}$. Find its Fourier series.

16. One can find the Fourier series for a function $f(x)$ in either the trigonometric or exponential forms:

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \left[a_k \frac{\cos kx}{\sqrt{\pi}} + b_k \frac{\sin kx}{\sqrt{\pi}} \right] \qquad \text{or} \qquad f(x) = \sum_{k=-\infty}^{\infty} c_k \frac{e^{ikx}}{\sqrt{2\pi}}.$$

If $f(x)$ is a real-valued function, by taking the real part of the exponential version, show how to obtain the trig version. NOTE: Recall Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

[Last revised: February 5, 2012]