Solution to Problem Set 8 #7

REMARK: The technique for Problem 11b) is similar. One uses the result from 11a) $u_x = (w_r)(x/r)$ noting that on the right side, x arises in three places: w_r, x, and r.

7. Say $u(x,t)$ is a solution of the one space dimensional *wave equation* $u_{tt} = u_{xx}$. Find a linear change of variable

$$\begin{aligned} x =& ar + bs, \\ t =& cr + ds \end{aligned}$$

so that $v(r,\, s) := u(ar + bs,\, cr + ds)$ satisfies

$$\frac{\partial v}{\partial r \partial s} = 0.$$

SOLUTION: By the chain rule:

$$v(r,s)_r = u_x x_r + u_t t_r = a u_x + c u_t. \tag{1}$$

We next want to take the partial derivative of this equation with respect to s. This must be done for both u_x and u_t. We first compute $\partial u_x(x(r,s),\, t(r,\, s))\big/\partial s$. It may help to let $w(x,\, t) := u_x(x,\, t)$. Then, just as in equation (1)

$$\frac{\partial w(x(r,s),\, t(r,\, s))}{\partial s} = w_x x_s + w_t t_s = b w_x + d w_t = b u_{xx} + d u_{xt}.$$

Similarly, with $z(x,t) := u_t(x,\, t)$

$$\frac{\partial z(x(r,s),\, t(r,\, s))}{\partial s} = z_x x_s + z_t t_s = b z_x + d z_t = b u_{tx} + d u_{tt}.$$

Consequently,

$$v(r,s)_{rs} = a[b u_{xx} + d u_{xt}] + c[b u_{tx} + d u_{tt}] = ab u_{xx} + (ad + bc) u_{xt} + cd u_{tt}.$$

We would like the right-hand side to be $u_{tt} - u_{xx}$, so

$$ab = -1, \qquad ad + bc = 0, \qquad cd = 1.$$

One solution is $a = c = d = 1$, $b = -1$. That is, the desired change of variables is

$$x = r - s, \qquad t = r + s.$$

[Last revised: March 9, 2012]