Problem Set 8

DUE: Never

Unless otherwise stated use the standard Euclidean *norm.*

1. Find a 3×3 symmetric matrix A with the property that

$$\langle X,\, AX\rangle = -x_1^2 + 4x_1x_2 - x_1x_3 + 2x_2x_3 + 5x_3^2$$

for all $X = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3$.

2. Let A and B be $n \times n$ real matrices. Show that $(AB)^* = B^*A^*$.

3. Consider the surface $x^2 + y^2 - 4z^2 = 1$.
   a) Find a vector orthogonal to the tangent plane at the point $(1,\, 2,\, -1)$.
   b) Find the equation of the tangent plane to this surface at this point.

4. a) Let $u(x)$ be a smooth solution of $u'' + 17u'(x) - c(x)u = 0$ for $x \in \mathbb{R}$ and assume that $c(x) > 0$ everywhere. Show that $u(x)$ cannot have a positive local maximum at any $x \in \mathbb{R}$.
   b) Let $u(x,y)$ be a smooth solution of $u_{xx} + 3u_{yy} - c(x,y)u = 0$ for $(x,y) \in \mathbb{R}^2$, where $c(x,y) > 0$ everywhere. Show that $u(x,y)$ cannot have a positive local maximum at any $(x,y) \in \mathbb{R}^2$.

5. Find a particular solution $u(x,y)$ of the inhomogeneous equation $u_x + 5u_y = 1 + 4x^2$.

6. Find some function $u(x,y)$ that satisfies $u_{xy} = \cos(x + 2y) - 2xy$.

7. Say $u(x,t)$ is a solution of the one space dimensional *wave equation* $u_{tt} = u_{xx}$. Find a linear change of variable

$$x = ar + bs,$$
$$t = cr + ds$$

so that $v(r,\, s) := u(ar + bs,\, cr + ds)$ satisfies

$$\frac{\partial^2 v}{\partial r \partial s} = 0.$$

8. Say the equation $f(X) := f(x, y, z) = 0$ implicitly defines a smooth surface in $\mathbb{R}^3$ (an example is the ellipsoid $x^2 + y^2 + 4z^2 = 4$). Let $P \in \mathbb{R}^3$ be a point *not* on this surface. Assume Q is an interior point on the surface that is closest to P. Show that the vector from P to Q is orthogonal to the tangent plane to the surface at Q.

   [SUGGESTION: Let $X(t)$ be a smooth curve in the surface with $X(0) = Q$. Then Q is the point on the curve that is closest to P.]

9. Let $f(s)$ be a smooth function of the real variable s and let $u(x, t) := f(x + 3t)$. Show that u satisfies the homogeneous partial differential equation $u_t - 3u_x = 0$.

10. Let $f(s)$ be a twice differentiable function of the real variable s and let $u(x, t) := f(ax + by)$, where a and b are any real numbers. Show that u satisfies the nonlinear Monge-Ampere partial differential equation $u_{xx}u_{yy} - (u_{xy})^2 = 0$.

11. In the plane $\mathbb{R}^2$, let $r^2 = x^2 + y^2$ so r is the distance from $(x,\, y)$ to the origin and assume that the function $u(x, y)$ depends only on r. Thus $u(x, y) = w(r)$ for some (smooth) function w.

    a) Show that $\dfrac{\partial u}{\partial x} = \dfrac{dw}{dr}\dfrac{x}{r}$.

    b) Show that
    $$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 w}{dr^2}\left(\frac{x^2}{r^2}\right) + \left(\frac{1}{r} - \frac{x^2}{r^3}\right)\frac{dw}{dr}.$$

    c) Show that
    $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}.$$

    d) Use the previous part to find all functions $u(x, y)$ depending only on r that satisfy the Laplace equation: $u_{xx} + u_{yy} = 0$.

[Last revised: March 8, 2012]