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# Area of tthe Sphere in $\mathbb{R}^n$

Let $A_{n-1}$ denote the "area" of the unit $n-1$ dimensional sphere $S^{n-1}$ in $\mathbb{R}^n$. We seek an inductive formula for $A_{n-1}$ using the observation that

$$\pi^{n/2} = \left(\int_{\mathbb{R}} e^{-t^2}\,dt\right)^n = \iint_{R^n} e^{-r^2} r^{n-1}\,dr\,dA_{n-1} = A_{n-1}J_n, \tag{1}$$

where

$$J_n := \int_0^\infty e^{-r^2} r^{n-1}\,dr.$$

Integrating this by parts with $u = r^{n-2}$ and $dv = e^{-r^2} r\,dr$ we find

$$J_n = \frac{n-2}{2} J_{n-2}. \tag{2}$$

But from (1), replacing $n$ by $n-2$ we have

$$\pi^{(n-2)/2} = A_{n-3}J_{n-2}. \tag{3}$$

Dividing (1) by (3) and using equation (2)

$$\pi = \frac{A_{n-1}\,J_n}{A_{n-3}\,J_{n-2}} = \frac{n-2}{2}\frac{A_{n-1}}{A_{n-3}}.$$

Thus,

$$A_n = \frac{2\pi}{n-1} A_{n-2}.$$

Now inductively use $A_1 = 2\pi$ and $A_2 = 4\pi$ to compute $A_3$ etc.

REMARK: If $n \ge 8$ then $\frac{2\pi}{n-1} < 1$, so it is clear that $\lim_{n\to\infty} A_n = 0$.

To compute the volume $V_n(R)$ of the ball of radius $R$, use that

$$\frac{dV_n(R)}{dR} = A_{n-1}(R) = R^{n-1}A_{n-1}.$$

[Last revised: April 23, 2012]