Math 260, Spring 2012 Jerry L. Kazdan

Matrices as Maps

We now discuss viewing systems of equations as maps. Think of this an an introduction to *computer graphics*. We'll use these ideas throughout Math 260.

The standard technique goes back to Descartes' introduction of coordinates in geometry. Say one has two copies of the plane, the first with coordinates (x_1, x_2), the second with coordinates (y_1, y_2). Then the high school equations

$$\begin{aligned} ax_1 + bx_2 &= y_1 \\ cx_1 + dx_2 &= y_2 \end{aligned} \tag{1}$$

can be thought of as a mapping from the (x_1, x_2) plane to the (y_1, y_2) plane. For instance, if $x_1 = 1$ and $x_2 = 0$, then $y_1 = a$ and $y_2 = c$. Thus the point $(1, 0)$ is mapped to the point (a, c). Similarly, the point $(0, 1)$ is mapped to (b, d).

Since the coefficients in these equations contain all the useful information, it is useful to collect them in a box as a *matrix*

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{and also write} \quad x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Now write the equations (1) using the shorthand

$$Ax = y. \tag{2}$$

So far, this is just notation. Imagine solving (1) for (x_1, x_2) in terms of (y_1, y_2), say

$$\begin{aligned} x_1 &= \alpha y_1 + \beta y_2 \\ x_2 &= \gamma y_1 + \delta y_2. \end{aligned} \tag{3}$$

This is the *inverse mapping* to $Ax = y$ so we write it as A^{-1}, that is, $x = A^{-1}y$, where

$$A^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

To actually compute the *inverse matrix*, A^{-1}, we simply use high school techniques to solve (1) in the form (3) and then read-off the coefficients.

Exercise If $A := \left(\begin{smallmatrix} 0 & 2 \\ -1 & 1 \end{smallmatrix}\right)$, use this approach to compute A^{-1}.

There are two particular maps that are very simple, yet fundamental: The *identity map* which leaves each point unchanged:

$$\begin{aligned} x_1 \quad &= y_1 \\ x_2 &= y_2 \end{aligned} \qquad \text{with matrix} \qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the *zero map* which maps *all* points (x_1, x_2) to the origin:

$$\begin{aligned} 0x_1 + 0x_2 &= y_1 \\ 0x_1 + 0x_2 &= y_2 \end{aligned} \qquad \text{with matrix} \qquad 0 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next we'll introduce a small yet valuable new idea. Say we have a second set of equations

$$\begin{aligned} ey_1 + fy_2 &= z_1 \\ gy_1 + hy_2 &= z_2 \end{aligned} \tag{4}$$

which we write for short in the matrix form

$$By = z, \qquad \text{where} \quad B := \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

We can view A as a map from x to y and B as a map from y to z. Combining them, that is, composing the maps, we obtain a map C from x to z by the rule

$$z = Cx := B(Ax),$$

and regard C as the *product*: $C = BA$. To compute C we substitute (1) into (4)

$$\begin{aligned} e(ax_1 + bx_2) + f(cx_1 + dx_2) =& z_1 \\ g(ax_1 + bx_2) + h(cx_1 + dx_2) =& z_2. \end{aligned}$$

Collecting terms yields

$$\begin{aligned} (ea + fc)x_1 + (eb + fd)x_2 =& z_1 \\ (ga + hc)x_1 + (gb + hd)x_2 =& z_2. \end{aligned}$$

Thus, we see that

$$BA = C \quad \text{means that} \quad \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

This is the complete motivation for the definition of *matrix multiplication*.

Example The product AA^{-1} makes sense and gives, as one might anticipate, $AA^{-1} = I$. Similarly, $A^{-1}A = I$.

Exercise If A and S are 2×2 matrices with S invertible, show that $(SAS^{-1})^2 = SA^2S^{-1}$.

[Last revised: January 3, 2012]