Intuition and Functions of Several Variables

Undergraduates are usually surprised to see the classical example in the plane $\mathbb{R}^2$,

$$f(x,y) = (y - x^2)(y - 2x^2),$$

that has a local minumum approaching the origin along straight lines, but the origin is actually a saddle point. The complicating ingredient is that in this example, the origin is a degenerate critical point (the second derivative matrix is not invertible). Approaching a non-degenerate critical point along straight lines does determine the nature of the critical point.

Long ago, in a distant galaxy I was asked to referee a proof of the positive mass conjecture in general relativity. The conjecture, eventually proved correct, was then an open question. The author's argument essentially was the following. He had a functional of the form

$$J(u) = \iiint_{\mathbb{R}^3} F(u, \nabla u)\, dx\, dy\, dz$$

where $F(u,p)$ was a smooth real-valued function of its variables and $u(x,y,z)$ lies in some reasonable function space. Since in his example $F(0,0) = 0$, it was clear that $J(0) = 0$. The goal was to prove that $J(u) \geq 0$ for all u. The author showed that $u(x,y,z) \equiv 0$ was the *only* critical point of $J(u)$, and also that it was a non-generate local minimum. Using intuition that was "obvious" to him, he then concluded that $u(x) \equiv 0$ was the *global* minimum of J, thus proving the desired inequality.

The point of this note is to give an example showing this intuition is false, even for fairly simple functions $f(x,y)$ of two variables [it is true for functions of one variable]. I will also give a few related examples.

EXAMPLE 1. To show how to arrive at the example in a few steps revealing the geometric idea, begin with

$$h(s,t) := s^3 - 3s + t^2,$$

which has two (non-degenerate) critical points, a saddle at $(-1,0)$ and a local min at $(1,0)$.

Make a change of variables to put the min in the upper half-plane and the saddle in the lower half-plane. A rotation by $\pi/4$ will do but the resulting formula is simpler using: $s = u$, $t = u - v$. In these new variables the example is

$$g(u,v) := u^3 - 3u + (u - v)^2$$

whose critical points are now at $(-1,-1)$ and $(1,1)$.

Finally, get rid of the saddle at $(-1,-1)$ by using a diffeomorphism of the whole plane onto the upper half-plane: $x = u$, $y = e^v$. The example is then

$$f(x,y) := x^3 - 3x + (x - e^y)^2.$$

It has a non-degenerate local min at $(1,0)$ and no other critical points. This not the global minimum since as $x \to -\infty$ then $f(x,y) \to -\infty$.

EXAMPLE 2. Several years ago I mentioned this question to my advanced calculus course and asked if they could find a polynomial example. The next week one student, James Pender, found the surprisingly simple polynomial

$$p(x,y) := (1+y)^3x^2 + y^2.$$

It has only one critical point (at the origin) where there is a non-degenerate local minimum. However this is *not* the global minimum.

A picture would really help. I used Maple to graph this and related examples, such as $g(x,y) := (1-y^2)^3x^2 + y^2$. None of my pictures satisfied me. Surely others can do better.

These examples are not very complicated. They show that the phenomenon is not pathological.

MORAL: Geometric intuition for functions of several variables can be difficult.

EXERCISE Find a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ that has exactly two local minima (non-degenerate) but no other critical points.

GENERALIZATION. The geometric ideas used in Example 1 can be used to show the well-known fact that given and non-negative integers M, s, and m there is a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ having exactly M local maxima, s saddle points, and m local minima, all non-degenerate, and no other critical points. Thus there can't be any Morse inequalities.

To construct an example, take a function g with infinitely many non-degenerate maxima, saddles, and minima, for instance $g(x,y) := \sin \pi x \sin \pi y$. Pick the required number of maxima, saddles, and minima, Let Γ be a smooth curve with no self-intersections that passes through these critical points and no others. Then let Ω be a tubular neighborhood of Γ that contains no additional critical points. There is a diffeomorphism $\varphi : \mathbb{R}^2 \to \Omega$. The desired example is $f := g \circ \varphi$.

This note was written backwards. I knew the construction just above first and when I needed it to referee the paper mentioned in the introduction, simply adapted the general argument to obtain Example 1.