Math 312 **Exam 1** Jerry L. Kazdan
Oct. 11, 2012 12:00 – 1:20

DIRECTIONS This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 5 problems (15 points each, so total is 75 points). Maximum score is thus 125 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (10 points each, so 50 points).

A–1. Which of the following sets are linear spaces? [If not, why not?]

a) The points $\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0$.

b) The set of points $(x, y) \in \mathbb{R}^2$ with $y = x^2$.

c) In $\mathbb{R}^2$, the span of the linearly dependent vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

d) The set of solutions $\vec{x}$ of $A\vec{x} = 0$, where A is a 4×3 matrix.

e) The set of polynomials $p(x)$ of degree at most 2 with $p'(1) = 0$.

A–2. Let $\mathcal{S}$ be the linear space of 2×2 matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a + d = 0$. Find a basis and compute the dimension of $\mathcal{S}$.

A–3. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say $\vec{v}_1$ and $\vec{v}_2$ are (distinct!) solutions of the equations $L\vec{x} = \vec{y}_1$ while $\vec{w}$ is a solution of $L\vec{x} = \vec{y}_2$. Answer the following in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$.

a) Find some solution of $L\vec{x} = 2\vec{y}_1 - 2\vec{y}_2$.

b) Find another solution (other than $\vec{w}$) of $L\vec{x} = \vec{y}_2$.

A–4. Say you have a matrix A.

a) If $A : \mathbb{R}^5 \to \mathbb{R}^5$, what are the possible dimensions of the kernel of A? The image of A?

b) If $B : \mathbb{R}^5 \to \mathbb{R}^3$, what are the possible dimensions of the kernel of B? The image of B?

A–5. Let A be *any* 5×3 matrix so $A\vec{x} : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation. Answer the following **with a brief explanation**.

a) Is $A\vec{x} = \vec{b}$ necessarily solvable for any $\vec{b}$ in $\mathbb{R}^5$?

b) Suppose the kernel of A is one dimensional. What is the dimension of the image of A?

PART B Five questions, 15 points each (so 75 points total).

B–1. Let $Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. [NOTE: In this problem, there is *no partial credit* for sloppy computations.]

a) Find the inverse of Q.

b) Find the inverse of Q^2.

B–2. Define the linear maps A, B, and C from $\mathbb{R}^2 \to \mathbb{R}^2$ by the rules

- A rotates vectors by $\pi/2$ radians counterclockwise.
- B reflects vectors across the horizontal axis.
- C orthogonal projection onto the vertical axis, so $(x_1, x_2) \to (0, x_2)$

Let M be the linear map that first applies A, then B, and finally C. Find a matrix that represents M in the standard basis for $\mathbb{R}^2$.

B–3. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$ be given matrices.

a) Show that $BA : \mathbb{R}^3 \to \mathbb{R}^3$ *cannot* be invertible.

b) Give an example where the matrix $AB : \mathbb{R}^2 \to \mathbb{R}^2$ *is* invertible.

B–4. a) Find all matrices $A : \mathbb{R}^3 \to \mathbb{R}^2$ whose kernels contain the vector $\vec{x} := \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

b) Find a basis for the linear space of these matrices.

B–5. Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map that send a polynomial $p(x)$ (of degree at most 2) to $p''(x) + 3p(x)$.

a) Find the matrix representation $[L]_{\mathcal{B}}$ of L using the basis $\mathcal{B} = \{1, x, x^2\}$.

b) Find a basis for the kernel of L (you may use your matrix $[L]_{\mathcal{B}}$).

c) Find a basis for the image of L (you may use your matrix $[L]_{\mathcal{B}}$).

d) Is L invertible? Why or why not?.