Math 312 | Exam 1 | Jerry L. Kazdan
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Exam 1

DIRECTIONS This exam has two parts. Part A has shorter 5 questions, (10 points each so total 50 points) while Part B had 5 problems (15 points each, so total is 75 points). Maximum score is thus 125 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (10 points each, so 50 points).

A–1. Which of the following sets are linear spaces? [If not, why not?]

a) The points $\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0$.

SOLUTION: This is a linear space.

b) The set of points $(x, y) \in \mathbb{R}^2$ with $y = x^2$.

SOLUTION: This is **not** a linear space. The point $(1, 1)$ is in this set but $(2, 2)$ is not.

c) In $\mathbb{R}^2$, the span of the linearly dependent vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

SOLUTION: This is a linear space. It is the linear space of all points of the form $(c, -c)$ for any real scalar c. Geometrically, this is a straight line through the origin in the plane $\mathbb{R}^2$.

d) The set of solutions $\vec{x}$ of $A\vec{x} = 0$, where A is a 4×3 matrix.

SOLUTION: This is a linear space: the kernel of A.

e) The set of polynomials $p(x)$ of degree at most 2 with $p'(1) = 0$.

SOLUTION: This is a linear space since if $p'(1) = 0$ and $q'(1) = 0$, then both (cp) and $(p + q)$ have the same property.

A–2. Let $\mathcal{S}$ be the linear space of 2×2 matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a + d = 0$. Find a basis and compute the dimension of $\mathcal{S}$.

SOLUTION: Since $d = -a$, these matrices all have the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so the dimension is 3.

A–3. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say $\vec{v}_1$ and $\vec{v}_2$ are (distinct!) solutions of the equations $L\vec{x} = \vec{y}_1$ while $\vec{w}$ is a solution of $L\vec{x} = \vec{y}_2$. Answer the following in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$.

a) Find some solution of $L\vec{x} = 2\vec{y}_1 - 2\vec{y}_2$.

SOLUTION: $2\vec{v}_1 - 2\vec{w}$. Another is $2\vec{v}_2 - 2\vec{w}$.

b) Find another solution (other than $\vec{w}$) of $L\vec{x} = \vec{y}_2$.

SOLUTION: $\vec{v}_1 - \vec{v}_2 + \vec{w}$. More generally, $c(\vec{v}_1 - \vec{v}_2) + \vec{w}$ for any scalar c

A–4. Say you have a matrix A.

a) If $A : \mathbb{R}^5 \to \mathbb{R}^5$, what are the possible dimensions of the kernel of A? The image of A?

SOLUTION: $0, 1, \ldots, 5$ for both the image and kernel. The special cases $A = 0$ and $A = I$ illustrate the extremes.

b) If $B : \mathbb{R}^5 \to \mathbb{R}^3$, what are the possible dimensions of the kernel of B? The image of B?

SOLUTION: The image can have dimensions 0, 1, 2, or 3. The kernel can have dimension $2, \ldots, 5$.

A–5. Let A be *any* 5×3 matrix so $A\vec{x} : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation. Answer the following **with a brief explanation**.

a) Is $A\vec{x} = \vec{b}$ necessarily solvable for any $\vec{b}$ in $\mathbb{R}^5$?

SOLUTION: Since the image can have dimension at most 3 and the target has dimension 5, the map cannot be onto, so there are many vectors $\vec{b}$ for which $A\vec{x} = \vec{b}$ has no solution.

b) Suppose the kernel of A is one dimensional. What is the dimension of the image of A?

SOLUTION: $\dim(\text{image}(A)) = 3 - 1 = 2$.

PART B Five questions, 15 points each (so 75 points total).

B–1. Let $Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. [NOTE: In this problem, there is *no partial credit* for sloppy computations.]

a) Find the inverse of Q.

SOLUTION: By a routine computation (the matrix is upper triangular),

$$Q^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

b) Find the inverse of Q^2.

SOLUTION: The point of this was that it is simplest to use

$$Q^{-2} = (Q^{-1})^2 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

B–2. Define the linear maps A, B, and C from $\mathbb{R}^2 \to \mathbb{R}^2$ by the rules

- A rotates vectors by $\pi/2$ radians counterclockwise.
- B reflects vectors across the horizontal axis.
- C orthogonal projection onto the vertical axis, so $(x_1, x_2) \to (0, x_2)$

Let M be the linear map that first applies A, then B, and finally C. Find a matrix that represents M in the standard basis for $\mathbb{R}^2$.

SOLUTION: **method 1** $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so $M = CBA = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$

method 2

$$\begin{array}{ccccccc} (1,0) & \xrightarrow{A} & (0,1) & \xrightarrow{B} & (0,-1) & \xrightarrow{C} & (0,-1) \\ (0,1) & \xrightarrow{A} & (-1,0) & \xrightarrow{B} & (-1,0) & \xrightarrow{C} & (0,0) \end{array}$$

so $M = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$

B–3. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$ be given matrices.

a) Show that $BA : \mathbb{R}^3 \to \mathbb{R}^3$ *cannot* be invertible.

SOLUTION: Since $A : \mathbb{R}^3 \to \mathbb{R}^2$, then $\dim(\ker A) \geq 1$ so there is a $\vec{z} \in \mathbb{R}^3$, $\vec{z} \neq 0$ such that $A\vec{z} = 0$. Consequently $BA\vec{z} = 0$ so BA is not one-to-one. Thus it is not invertible.

b) Give an example where the matrix $AB : \mathbb{R}^2 \to \mathbb{R}^2$ *is* invertible.

SOLUTION: Essentially almost any A and B will give an example. Perhaps the simplest is

$$A : (x_1, x_2, x_3) \to (x_1, x_2) \qquad \text{and} \qquad B : (x_1, x_2) \to (x_1, x_2, 0).$$

As matrices, $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

B–4. a) Find all matrices $A : \mathbb{R}^3 \to \mathbb{R}^2$ whose kernels contain the vector $\vec{x} := \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

SOLUTION: Say $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. Then

$$0 = A\vec{x} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} a - 2c \\ d - 2f \end{pmatrix}$$

so $a = 2c$ and $d = 2f$. Thus

$$A = \begin{pmatrix} 2c & b & c \\ 2f & e & f \end{pmatrix}$$

for any scalars $b. c$, e, and f.

b) Find a basis for the linear space of these matrices.

SOLUTION: From the previous part

$$A = c\begin{pmatrix}2 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + f\begin{pmatrix}0 & 0 & 0\\ 2 & 0 & 1\end{pmatrix} + e\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

so the 4 matrices above are a basis.

B–5. Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map that send a polynomial $p(x)$ (of degree at most 2) to $p''(x) + 3p(x)$.

a) Find the matrix representation $[L]_{\mathcal{B}}$ of L using the basis $\mathcal{B} = \{1,\, x,\, x^2\}$.

SOLUTION: By a straightforward computation, if $p(x) = a + bx + cx^2$, then

$$Lp(x) = 2c + 3(a + bx + cx^2) = (3a + 2c)1 + 3bx + 3cx^2.$$

If $q(x) = \alpha + \beta x + \gamma x^2$, we can seek a polynomial $p(x) \in \mathcal{P}_2$ so that $Lp = q$. Comparing Lp and q above, we need to pick the coefficients a, b, and c so that

$$\begin{aligned} 3a + 2c &= \alpha \\ 3b &= \beta \\ 3c &= \gamma \end{aligned} \qquad \text{that is,} \qquad \begin{pmatrix}3 & 0 & 2\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}. \tag{1}$$

The 3×3 matrix above is the desired matrix $[L]_{\mathcal{B}}$ of L in the basis $\mathcal{B} = \{1,\, x,\, x^2\}$. Let's do this more formally. First we explicitly introduce the basis

$$e_1(x) := 1, \qquad e_2(x) := x, \qquad e_3(x) = x^2.$$

Then

$$p(x) = ae_1(x) + be_2(x) + ce_3(x)$$

and

$$Le_1 = 3 = 3e_1, \qquad Le_2 = 3x = 3e_2, \qquad Le_3 = 2 + 3x^2 = 2e_1 + 3e_3.$$

This gives the 3 columns of the matrix on the right in (1)

b) Find a basis for the kernel of L (you may use your matrix $[L]_{\mathcal{B}}$).

SOLUTION: Either solve the there equations

$$\begin{aligned} 3a + 2c &= 0 \\ 3b &= 0 \\ 3c &= 0 \end{aligned}$$

or find the kernel of the matrix $\begin{pmatrix}3 & 0 & 2\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}$. Both instantly show that the kernel of L *acting on quadratic polynomials* only has $p(x) = 0$.

c) Find a basis for the image of L (you may use your matrix $[L]_{\mathcal{B}}$).

SOLUTION: Since $[L]_{\mathcal{B}}$ is a square matrix whose kernel is trivial (only the vector representing the polynomial $p(x) \equiv 0$), it is invertible, so any basis for $\mathcal{P}_2$, such as $\{1, x, x^2\}$ is a basis for the image.

d) Is L invertible? Why or why not?.

SOLUTION: It is invertible. See the answer in part c).