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Math 312 **Exam 2** Jerry L. Kazdan
Nov. 15, 2012 12:00 – 1:20

DIRECTIONS This exam has two parts. Part A has 6 shorter questions, (5 points each so total 30 points) while Part B had 5 problems (10 points each, so total is 50 points). Maximum score is thus 80 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Six short answer questions (5 points each, so 30 points). To receive credit you *must* explain your reasoning at least briefly.

A–1. Find all *invertible* linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$ with the property $A^2 = 3A$.

A–2. Let $A : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear map. If dim im(A) = 2, what is $\dim \operatorname{im}(A)^\perp$?

A–3. If a certain matrix C satisfies $\langle \vec{x}, C\vec{y}\rangle = 0$ for *all* vectors $\vec{x}$ and $\vec{y}$, show that $C = 0$.

A–4. Say $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with the property that $A^2 - 3A + 2I = 0$. If $\vec{v} \neq 0$ is an eigenvector of A with eigenvalue λ,, so $A\vec{v} = \lambda\vec{v}$, what are the possible values of λ?

A–5. Under what conditions on the constants a, b, c, and d is the following matrix A positive definite, that is, $\langle \vec{x}, A\vec{x}\rangle > 0$ for all $\vec{x} \neq 0$?

$$A := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

A–6. Let A be an $n \times n$ matrix with columns $A_1, A_2, \ldots, A_n$ and let B be the matrix where A_1 (the first column of A), is replaced by $3A_1 + A_2$ and the other columns are unchanged. Compute $\det B$ in terms of $\det A$.

PART B: Five problems (10 points each, so 50 points).

B–1. Consider the space of real cubic polynomials $\mathcal{P}_3$ having degree at most 3, so $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with the inner product $\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)x^2\,dx$ [NOTE: This is *not* the usual inner product.]

Find the orthogonal projection of x^3 into the subspace spanned by 1 and x.

B–2. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Say $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

a) Compute $A^2\vec{x}$ and $A^3\vec{x}$ in terms of the c_i, λ_i and $\vec{v}_i$, $i = 1, \ldots, n$.

b) If $\lambda_1 = 1$ and the remaining λ_j satisfy $|\lambda_j| < 1$, $j = 2, \ldots, n$, compute $\lim_{k\to\infty} A^k\vec{x}$. [This arises in the study of *Markov Chains*].

B–3. In an experiment, at time t you measure the value of a quantity R and obtain:

t	-1	0	1	2
R	-1	1	1	-3

Based on other information, you believe the data should fit a curve of the form $R = a+bt^2$.

a) Write the (over-determined) system of linear equations you would ideally like to solve for the unknown coefficients a and b.

b) Use the method of least squares to find the *normal equations* for the coefficients a and b.

c) Solve the normal equations to find the coefficients a and b *explicitly* (numbers, like 3/5 and -2).

B–4. A projection $P : \mathbb{R}^n \to \mathbb{R}^n$ (so $P^2 = P$) is called an *orthogonal projection* if the image of P and kernel of P are orthogonal subspaces.

If P is self-adjoint, so $P^* = P$, show that P is an orthogonal projection. [REMARK: The converse is also true: If P is an orthogonal projection, then $P = P^*$. You are not asked to prove this here.]

B–5. Let A be a real $n \times n$ anti-symmetric matrix, so $A^* = -A$.

a) Show that $A\vec{x}$ is orthogonal to $\vec{x}$ for *every* vector $\vec{x}$.

b) Say $\vec{x}(t)$ is a solution of the differential equation $\dfrac{d\vec{x}}{dt} = A\vec{x}$, where A is an anti-symmetric matrix. Show that $\|\vec{x}(t)\| = \text{constant}$.