determine the optimal sustainable yield in each case. Make sure that you only allow k to take on integer values in your calculations.

(c) Repeat the calculations in part (b) using

$$\rho = 1.91, 1.92, 1.93, 1.94, 1.95, 1.96, 1.97, 1.98, 1.99$$

(d) Show that if $\rho = 2$, then the optimal sustainable yield can never be larger than $2as$.

(e) Compare the values of k determined in parts (b) and (c) to $1/(2-\rho)$, and use some calculus to explain why

$$k \simeq \frac{1}{2-\rho}$$

T2. A particular forest has growth parameters given by

$$g_i = \frac{1}{2^i}$$

for $i = 1, 2, 3, \ldots, n-1$, where n (the total number of height classes) can be chosen as large as needed. Suppose that the value of a tree in the kth height interval is given by

$$p_k = a(k-1)^\rho$$

where a is a constant (in dollars) and ρ is a parameter satisfying $1 \le \rho$.

(a) Show that the yield Yld_k is given by

$$Yld_k = \frac{a(k-1)^\rho s}{2^k - 2}$$

(b) For

$$\rho = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

use a computer to determine the class number that should be completely harvested in order to obtain an optimal yield, and determine the optimal sustainable yield in each case. Make sure that you only allow k to take on integer values in your calculations.

(c) Compare the values of k determined in part (b) to $1 + \rho/\ln(2)$ and use some calculus to explain why

$$k \simeq 1 + \frac{\rho}{\ln(2)}$$

11.11 COMPUTER GRAPHICS

In this section we assume that a view of a three-dimensional object is displayed on a video screen and show how matrix algebra can be used to obtain new views of the object by rotation, translation, and scaling.

PREREQUISITES: Matrix Algebra
Analytic Geometry

Visualization of a Three-Dimensional Object Suppose that we want to visualize a three-dimensional object by displaying various views of it on a video screen. The object we have in mind to display is to be determined by a finite number of straight line segments. As an example, consider the truncated right pyramid with hexagonal base illustrated in Figure 11.11.1. We first introduce an xyz-coordinate system in which to embed the object. As in Figure 11.11.1, we orient the coordinate system so that its origin is at the center of the video screen and the xy-plane coincides with the plane of the screen. Consequently, an observer will see only the projection of the view of the three-dimensional object onto the two-dimensional xy-plane.

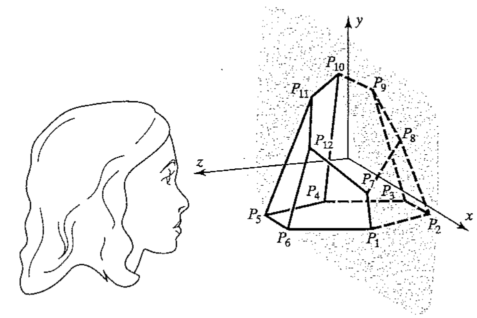


Figure 11.11.1

In the xyz-coordinate system, the endpoints $P_1, P_2, \ldots, P_n$ of the straight line segments that determine the view of the object will have certain coordinates, say,

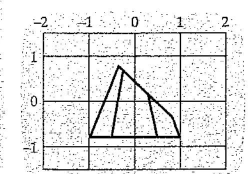
$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad \ldots, \quad (x_n, y_n, z_n)$$

These coordinates, together with a specification of which pairs are to be connected by straight line segments, are to be stored in the memory of the video display system. For example, assume that the 12 vertices of the truncated pyramid in Figure 11.11.1 have the following coordinates (the screen is 4 units wide by 3 units high):

$$\begin{array}{ll}
P_1: (1.000, -.800, .000), & P_2: (.500, -.800, -.866), \\
P_3: (-.500, -.800, -.866), & P_4: (-1.000, -.800, .000), \\
P_5: (-.500, -.800, .866), & P_6: (.500, -.800, .866), \\
P_7: (.840, -.400, .000), & P_8: (.315, .125, -.546), \\
P_9: (-.210, .650, -.364), & P_{10}: (-.360, .800, .000), \\
P_{11}: (-.210, .650, .364), & P_{12}: (.315, .125, .546)
\end{array}$$

These 12 vertices are connected pairwise by 18 straight line segments as follows, where $P_i \leftrightarrow P_j$ denotes that point P_i is connected to point P_j:

$$\begin{array}{llllll}
P_1 \leftrightarrow P_2, & P_2 \leftrightarrow P_3, & P_3 \leftrightarrow P_4, & P_4 \leftrightarrow P_5, & P_5 \leftrightarrow P_6, & P_6 \leftrightarrow P_1, \\
P_7 \leftrightarrow P_8, & P_8 \leftrightarrow P_9, & P_9 \leftrightarrow P_{10}, & P_{10} \leftrightarrow P_{11}, & P_{11} \leftrightarrow P_{12}, & P_{12} \leftrightarrow P_7, \\
P_1 \leftrightarrow P_7, & P_2 \leftrightarrow P_8, & P_3 \leftrightarrow P_9, & P_4 \leftrightarrow P_{10}, & P_5 \leftrightarrow P_{11}, & P_6 \leftrightarrow P_{12}
\end{array}$$



View 1

In View 1 these 18 straight line segments are shown as they would appear on the video screen. It should be noticed that only the x- and y-coordinates of the vertices are needed by the video display system to draw the view, as only the projection of the object onto the xy-plane is displayed. However, we must keep track of the z-coordinates to carry out certain transformations discussed later.

We now show how to form new views of the object by scaling, translating, or rotating the initial view. We first construct a $3 \times n$ matrix P, referred to as the *coordinate matrix of the view*, whose columns are the coordinates of the n points of a view:

$$P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix}$$

For example, the coordinate matrix P corresponding to View 1 is the 3×12 matrix

$$\begin{bmatrix} 1.000 & .500 & -.500 & -1.000 & -.500 & .500 & .840 & .315 & -.210 & -.360 & -.210 & .315 \\ -.800 & -.800 & -.800 & -.800 & -.800 & -.800 & -.400 & .125 & .650 & .800 & .650 & .125 \\ .000 & -.866 & -.866 & .000 & .866 & .866 & .000 & -.546 & -.364 & .000 & .364 & .546 \end{bmatrix}$$

We will show below how to transform the coordinate matrix P of a view to a new coordinate matrix P' corresponding to a new view of the object. The straight line segments connecting the various points move with the points as they are transformed. In this way each view is uniquely determined by its coordinate matrix once we have specified which pairs of points in the original view are to be connected by straight lines.

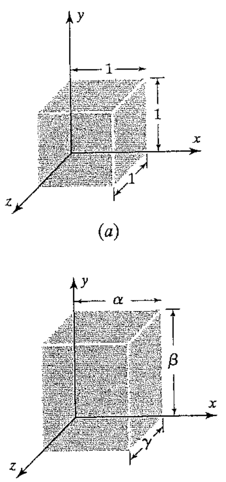


Figure 11.11.2

Scaling The first type of transformation we consider consists of scaling a view along the x, y, and z directions by factors of α, β, and γ, respectively. By this we mean that if a point P_i has coordinates (x_i, y_i, z_i) in the original view, it is to move to a new point P_i' with coordinates $(\alpha x_i, \beta y_i, \gamma z_i)$ in the new view. This has the effect of transforming a unit cube in the original view to a rectangular parallelepiped of dimensions $\alpha \times \beta \times \gamma$ (Figure 11.11.2). Mathematically, this may be accomplished with matrix multiplication as follows. Define a 3×3 diagonal matrix

$$S = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

Then, if a point P_i in the original view is represented by the column vector

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

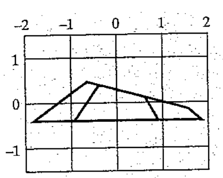
the transformed point P_i' is represented by the column vector

$$\begin{bmatrix} x_i' \\ y_i' \\ z_i' \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

Using the coordinate matrix P, which contains the coordinates of all n points of the original view as its columns, these n points can be transformed simultaneously to produce the coordinate matrix P' of the scaled view, as follows:

$$SP = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix}$$

$$= \begin{bmatrix} \alpha x_1 & \alpha x_2 & \cdots & \alpha x_n \\ \beta y_1 & \beta y_2 & \cdots & \beta y_n \\ \gamma z_1 & \gamma z_2 & \cdots & \gamma z_n \end{bmatrix} = P'$$


View 2
View 1 scaled by
$\alpha = 1.8, \beta = 0.5, \gamma = 3.0$.

The new coordinate matrix can then be entered into the video display system to produce the new view of the object. As an example, View 2 is View 1 scaled by setting $\alpha = 1.8$, $\beta = 0.5$, and $\gamma = 3.0$. Notice that the scaling $\gamma = 3.0$ along the z-axis is not visible in View 2, since we see only the projection of the object onto the xy-plane.

Translation We next consider the transformation of translating or displacing an object to a new position on the screen. Referring to Figure 11.11.3, suppose we desire

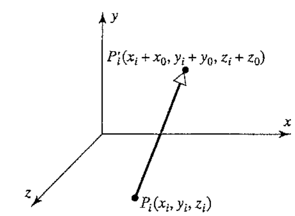


Figure 11.11.3

to change an existing view so that each point P_i with coordinates (x_i, y_i, z_i) moves to a new point P_i' with coordinates $(x_i + x_0, y_i + y_0, z_i + z_0)$. The vector

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

is called the ***translation vector*** of the transformation. By defining a $3 \times n$ matrix T as

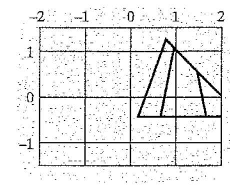
$$T = \begin{bmatrix} x_0 & x_0 & \cdots & x_0 \\ y_0 & y_0 & \cdots & y_0 \\ z_0 & z_0 & \cdots & z_0 \end{bmatrix}$$

all n points of the view determined by the coordinate matrix P can be translated by matrix addition by means of the equation

$$P' = P + T$$

The coordinate matrix P' then specifies the new coordinates of the n points. For example, if we wish to translate View 1 according to the translation vector

$$\begin{bmatrix} 1.2 \\ 0.4 \\ 1.7 \end{bmatrix}$$


View 3
View 1 translated by
$x_0 = 1.2, y_0 = 0.4, z_0 = 1.7$.

the result is View 3. Notice, again, that the translation $z_0 = 1.7$ along the z-axis does not show up explicitly in View 3.

In Exercise 7, a technique of performing translations by matrix multiplication rather than by matrix addition is explained.

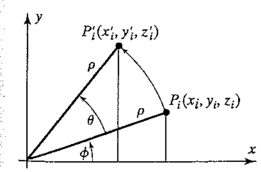


Figure 11.11.4

Rotation A more complicated type of transformation is a rotation of a view about one of the three coordinate axes. We begin with a rotation about the z-axis (the axis perpendicular to the screen) through an angle θ. Given a point P_i in the original view with coordinates (x_i, y_i, z_i), we wish to compute the new coordinates (x_i', y_i', z_i') of the rotated point P_i'. Referring to Figure 11.11.4 and using a little trigonometry, the reader should be able to derive the following:

$$x_i' = \rho\cos(\phi + \theta) = \rho\cos\phi\cos\theta - \rho\sin\phi\sin\theta = x_i\cos\theta - y_i\sin\theta$$
$$y_i' = \rho\sin(\phi + \theta) = \rho\cos\phi\sin\theta + \rho\sin\phi\cos\theta = x_i\sin\theta + y_i\cos\theta$$
$$z_i' = z_i$$

These equations can be written in matrix form as

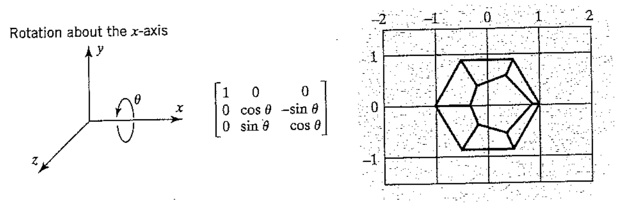
$$\begin{bmatrix} x_i' \\ y_i' \\ z_i' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

If we let R denote the 3×3 matrix in this equation, all n points can be rotated by the matrix product

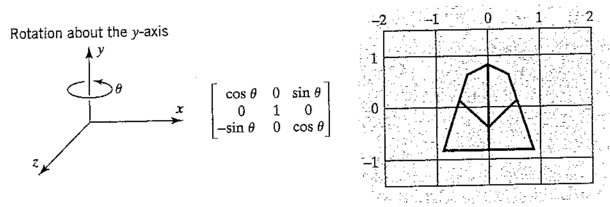
$$P' = RP$$

to yield the coordinate matrix P' of the rotated view.

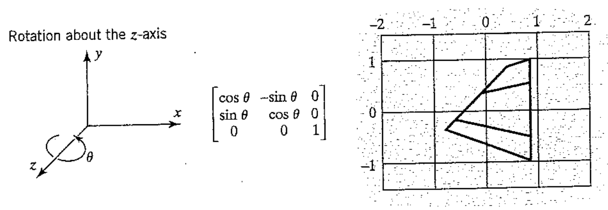
Rotations about the x and y axes can be accomplished analogously, and the resulting rotation matrices are given with Views 4, 5, and 6. These three new views of the truncated pyramid correspond to rotations of View 1 about the x, y, and z axes, respectively, each through an angle of 90°.



View 4 View 1 rotated 90° about the x-axis.

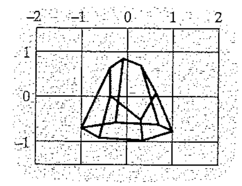


View 5 View 1 rotated 90° about the y-axis.



View 6 View 1 rotated 90° about the z-axis.

Rotations about three coordinate axes may be combined to give oblique views of an object. For example, View 7 is View 1 rotated first about the x-axis through 30°, then about the y-axis through −70°, and finally about the z-axis through −27°. Mathematically, these three successive rotations can be embodied in the single transformatic equation $P' = RP$, where R is the product of three individual rotation matrices:



View 7
Oblique view of truncated pyramid.

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(30°) & -\sin(30°) \\ 0 & \sin(30°) & \cos(30°) \end{bmatrix}$$

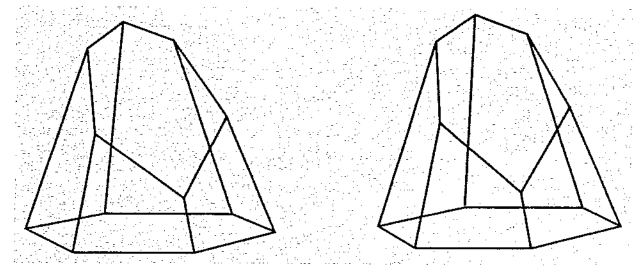
$$R_2 = \begin{bmatrix} \cos(-70°) & 0 & \sin(-70°) \\ 0 & 1 & 0 \\ -\sin(-70°) & 0 & \cos(-70°) \end{bmatrix}$$

$$R_3 = \begin{bmatrix} \cos(-27°) & -\sin(-27°) & 0 \\ \sin(-27°) & \cos(-27°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in the order

$$R = R_3R_2R_1 = \begin{bmatrix} .305 & -.025 & -.952 \\ -.155 & .985 & -.076 \\ .940 & .171 & .296 \end{bmatrix}$$

As a final illustration, in View 8 we have two separate views of the truncated pyramic which constitute a stereoscopic pair. They were produced by first rotating View 7 abou the y-axis through an angle of −3° and translating it to the right, then rotating the sam View 7 about the y-axis through an angle of +3° and translating it to the left. Th translation distances were chosen so that the stereoscopic views are about $2\frac{1}{2}$ inche apart—the approximate distance between a pair of eyes.



View 8 Stereoscopic figure of truncated pyramid. The three-dimensionality of the diagram can be seen by holding the book about one foot away and focusing on a distant object. Then by shifting gaze to View 8 without refocusing, the two views of the stereoscopic pair can be made to merge together and produce the desired effect.

Exercise Set 11.11

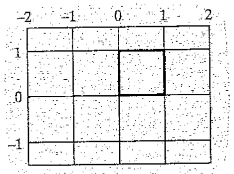
1. View 9 is a view of a square with vertices (0, 0, 0), (1, 0, 0), (1, 1, 0), and (0, 1, 0).
   (a) What is the coordinate matrix of View 9?
   (b) What is the coordinate matrix of View 9 after it is scaled by a factor $1\frac{1}{2}$ in the x-direction and $\frac{1}{2}$ in the y-direction? Draw a sketch of the scaled view.

(c) What is the coordinate matrix of View 9 after it is translated by the vector

$$\begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}?$$

Draw a sketch of the translated view.

(d) What is the coordinate matrix of View 9 after it is rotated through an angle of $-30°$ about the z-axis? Draw a sketch of the rotated view.



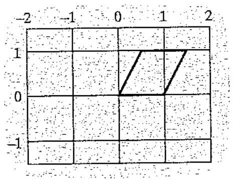
View 9 Square with vertices (0, 0, 0), (1, 0, 0), (1, 1, 0), and (0, 1, 0) (Exercises 1 and 2).

2. (a) If the coordinate matrix of View 9 is multiplied by the matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the result is the coordinate matrix of View 10. Such a transformation is called a *shear in the x-direction with factor* $\frac{1}{2}$ *with respect to the y-coordinate.* Show that under such a transformation, a point with coordinates (x_i, y_i, z_i) has new coordinates $(x_i + \frac{1}{2}y_i, y_i, z_i)$.

(b) What are the coordinates of the four vertices of the shear square in View 10?

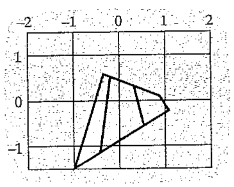


View 10 View 9 sheared along x-axis by $\frac{1}{2}$ with respect to the y-coordinate (Exercise 2).

(c) The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ .6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

determines a *shear in the y-direction with factor .6 with respect to the x-coordinate* (e.g., View 11). Sketch a view of the square in View 9 after such a shearing transformation, and find the new coordinates of its four vertices.

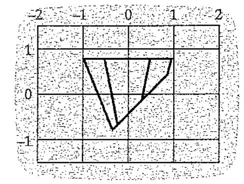


View 11 View 1 sheared along y-axis by .6 with respect to the x-coordinate (Exercise 2).

3. (a) The *reflection about the xz-plane* is defined as the transformation that takes a point (x_i, y_i, z_i) to the point $(x_i, -y_i, z_i)$ (e.g., View 12). If P and P' are the coordinate matrices of a view and its reflection about the xz-plane, respectively, find a matrix M such that $P' = MP$.

(b) Analogous to part (a), define the *reflection about the yz-plane* and construct the corresponding transformation matrix. Draw a sketch of View 1 reflected about the yz-plane.

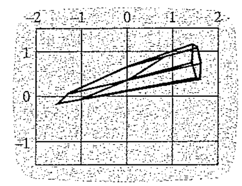
(c) Analogous to part (a), define the *reflection about the xy-plane* and construct the corresponding transformation matrix. Draw a sketch of View 1 reflected about the xy-plane.



View 12 View 1 reflected about the xz-plane (Exercise 3).

4. (a) View 13 is View 1 subject to the following five transformations:

1. Scale by a factor of $\frac{1}{2}$ in the x-direction, 2 in the y-direction, and $\frac{1}{3}$ in the z-direction.
2. Translate $\frac{1}{2}$ unit in the x-direction.
3. Rotate 20° about the x-axis.
4. Rotate $-45°$ about the y-axis.
5. Rotate 90° about the z-axis.



View 13 View 1 scaled, translated, and rotated (Exercise 4).

Construct the five matrices M_1, M_2, M_3, M_4, and M_5 associated with these five transformations.

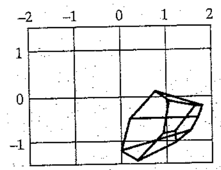
(b) If P is the coordinate matrix of View 1 and P' is the coordinate matrix of View 13, express P' in terms of M_1, M_2, M_3, M_4, M_5, and P.

5. (a) View 14 is View 1 subject to the following seven transformations:

1. Scale by a factor of .3 in the x-direction and .5 in the y-direction.
2. Rotate 45° about the x-axis.
3. Translate 1 unit in the x-direction.
4. Rotate 35° about the y-axis.
5. Rotate $-45°$ about the z-axis.
6. Translate 1 unit in the z-direction.
7. Scale by a factor of 2 in the x-direction.

Construct the matrices $M_1, M_2, \ldots, M_7$ associated with these seven transformations.

(b) If P is the coordinate matrix of View 1 and P' is the coordinate matrix of View 14, express P' in terms of $M_1, M_2, \ldots, M_7$, and P.



View 14 View 1 scaled, translated, and rotated (Exercise 5).

6. Suppose that a view with coordinate matrix P is to be rotated through an angle θ about an axis through the origin and specified by two angles α and β (see the accompanying figure). If P' is the coordinate matrix of the rotated view, find rotation matrices R_1, R_2, R_3, R_4, and R_5 such that

$$P' = R_5R_4R_3R_2R_1P$$

[***Hint.*** The desired rotation can be accomplished in the following five steps:

1. Rotate through an angle of β about the y-axis.
2. Rotate through an angle of α about the z-axis.
3. Rotate through an angle of θ about the y-axis.
4. Rotate through an angle of $-\alpha$ about the z-axis.
5. Rotate through an angle of $-\beta$ about the y-axis.]

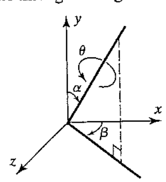


Figure Ex-6

7. This exercise illustrates a technique for translating a point with coordinates (x_i, y_i, z_i) to a point with coordinates $(x_i + x_0, y_i + y_0, z_i + z_0)$ by matrix multiplication rather than matrix addition.

(a) Let the point (x_i, y_i, z_i) be associated with the column vector

$$\mathbf{v}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

and let the point $(x_i + x_0, y_i + y_0, z_i + z_0)$ be associated with the column vector

$$\mathbf{v}'_i = \begin{bmatrix} x_i + x_0 \\ y_i + y_0 \\ z_i + z_0 \\ 1 \end{bmatrix}.$$

Find a 4×4 matrix M such that $\mathbf{v}'_i = M\mathbf{v}_i$.

(b) Find the specific 4×4 matrix of the above form that will effect the translation of the point $(4, -2, 3)$ to the point $(-1, 7, 0)$.

8. For the three rotation matrices given with Views 4, 5, and 6, show that

$$R^{-1} = R^T$$

(A matrix with this property is called an ***orthogonal matrix.*** See Section 6.5.)

Technology Exercises 11.11

The following exercises are designed to be solved using a technology utility. Typically, this will be MATLAB, *Mathematica*, Maple, Derive, or Mathcad, but it may also be some other type of linear algebra software or a scientific calculator with some linear algebra capabilities. For each exercise you will need to read the relevant documentation for the particular utility you are using. The goal of these exercises is to provide you with a basic proficiency with your technology utility. Once you have mastered the techniques in these exercises, you will be able to use your technology utility to solve many of the problems in the regular exercise sets.

T1. Let (a, b, c) be a unit vector normal to the plane $ax + by + cz = 0$, and let $\mathbf{r} = (x, y, z)$ be a vector. It can be shown that the mirror image of the vector $\mathbf{r}$ through the above plane has coordinates $\mathbf{r}_m = (x_m, y_m, z_m)$, where

$$\begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

with

$$M = I - 2\mathbf{n}\mathbf{n}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} a \\ b \\ c \end{bmatrix} [a \quad b \quad c]$$

(a) Show that $M^2 = I$ and give a physical reason why this must be so. [***Hint.*** Use the fact that (a, b, c) is a unit vector to show that $\mathbf{n}^T\mathbf{n} = 1$.]

(b) Use a computer to show that $\det(M) = -1$.

(c) The eigenvectors of M satisfy the equation

$$\begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and therefore correspond to those vectors whose direction is not affected by a reflection through the plane. Use a computer to determine the eigenvectors and eigenvalues of M and then give a physical argument to support your answer.

T2. A vector $\mathbf{v} = (x, y, z)$ is rotated by an angle θ about an axis having unit vector (a, b, c), thereby forming the rotated vector $\mathbf{v}_R = (x_R, y_R, z_R)$. It can be shown that

$$\begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

with

$$R(\theta) = \cos(\theta) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1 - \cos(\theta)) \begin{bmatrix} a \\ b \\ c \end{bmatrix} [a \quad b \quad c] + \sin(\theta) \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

(a) Use a computer to show that $R(\theta)R(\varphi) = R(\theta + \varphi)$, and then give a physical reason why this must be so. Depending on the sophistication of the computer you are using, you may have to experiment using different values of a, b, and

$$c = \sqrt{1 - a^2 - b^2}$$

(b) Show also that $R^{-1}(\theta) = R(-\theta)$ and explain physically why this must be so.
(c) Use a computer to show that $\det(R(\theta)) = +1$.

11.12 EQUILIBRIUM TEMPERATURE DISTRIBUTIONS

In this section we shall see that the equilibrium temperature distribution within a trapezoidal plate can be found when the temperatures around the edges of the plate are specified. The problem is reduced to solving a system of linear equations. Also, an interactive technique for solving the problem and a "random walk" approach to the problem are described.

PREREQUISITES: Linear Systems
Matrices
Intuitive Understanding of Limits

Boundary Data Suppose that the two faces of the thin trapezoidal plate shown in Figure 11.12.1*a* are insulated from heat. Suppose that we are also given the temperature along the four edges of the plate. For example, let the temperature be constant on each edge with values of 0°, 0°, 1°, and 2°, as in the figure. After a period of time the temperature inside the plate will stabilize. Our objective in this section is to determine this equilibrium temperature distribution at the points inside the plate. As we will see, the interior equilibrium temperature is completely determined by the ***boundary data***, that is, the temperature along the edges of the plate.

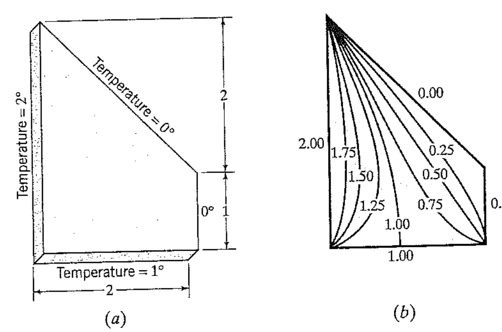


Figure 11.12.1

The equilibrium temperature distribution can be visualized by the use of curves that connect points of equal temperature. Such curves are called ***isotherms*** of the temperature distribution. In Figure 11.12.1*b* we have sketched a few isotherms using information we derive later in the chapter.

Although all our calculations will be for the trapezoidal plate illustrated, our techniques generalize easily to a plate of any practical shape. They also generalize to the problem of finding the temperature within a three-dimensional body. In fact, our "plate" could be the cross section of some solid object if the flow of heat perpendicular to the cross section is negligible. For example, Figure 11.12.1 could represent the cross section of a long dam. The dam is exposed to three different temperatures: the temperature of the ground at its base, the temperature of the water on one side, and the temperature of the air on the other side. A knowledge of the temperature distribution inside the dam is necessary to determine the thermal stresses to which it is subjected.

Next, we shall consider a certain thermodynamic principle that characterizes the temperature distribution we are seeking.

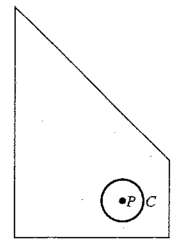


Figure 11.12.2

The Mean-Value Property There are many different ways to obtain a mathematical model for our problem. The approach we use is based on the following property of equilibrium temperature distributions.

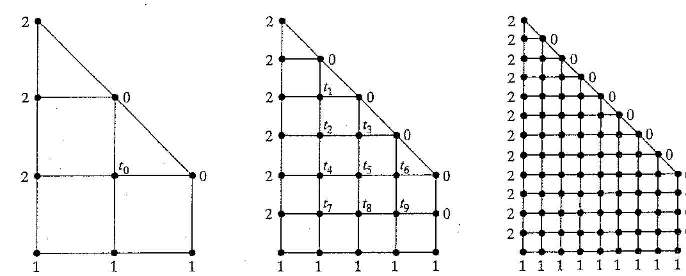
Theorem 11.12.1 The Mean-Value Property

Let a plate be in thermal equilibrium and let P be a point inside the plate. Then if C is any circle with center at P that is completely contained in the plate, the temperature at P is the average value of the temperature on the circle (Figure 11.12.2).

This property is a consequence of certain basic laws of molecular motion, and we will not attempt to derive it. Basically, this property states that in equilibrium, thermal energy tends to distribute itself as evenly as possible consistent with the boundary conditions. It can be shown that the mean-value property uniquely determines the equilibrium temperature distribution of a plate.

Unfortunately, determining the equilibrium temperature distribution from the mean-value property is not an easy matter. However, if we restrict ourselves to finding the temperature only at a finite set of points within the plate, the problem can be reduced to solving a linear system. We pursue this idea next.

Discrete Formulation of the Problem We can overlay our trapezoidal plate with a succession of finer and finer square nets or meshes (Figure 11.12.3). In (*a*) we have a rather coarse net; in (*b*) we have a net with half the spacing as in (*a*); and



(*a*) 1 interior mesh point (*b*) 9 interior mesh points (*c*) 49 interior mesh points

Figure 11.12.3