Math 312: Class meeting 10

Some applications

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September 26, 2012

Errors in transmission

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- Errors (0s and 1s being randomly swapped) will occur.
- Examples: WiFi, cell phones, lasers reading DVDs, etc.
- Problem: devise a scheme for errors to be detected, corrected automatically by receiver.

Naive solution 1

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- We'll ignore possibility of multiple errors.

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- The cost: requires 3 times as much data to send.

$(7, 4)$ Hamming code

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- "$(7,4)$ Hamming code" takes a 4 bit message, encodes it as 7 bits, requiring only $7/4 = 1.75$ times as much data.
- Can detect and correct single errors.
- Setup: say the senders message is x_1, x_2, x_3, x_4 (each 0 or 1). Denote by the vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

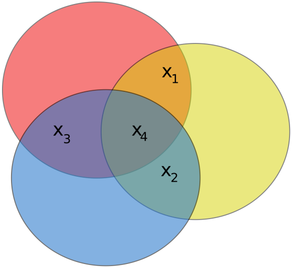
Parity

- We'll use addition "mod 2", meaning:

$$
\begin{aligned}
0 + 0 &= 0 \\
0 + 1 &= 1 \\
1 + 0 &= 1 \\
1 + 1 &= 0,
\end{aligned}
$$

How it works

- Write the bits in a Venn diagram:



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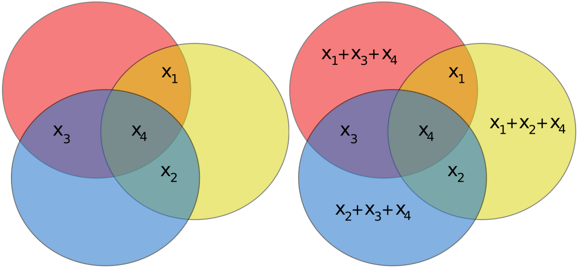
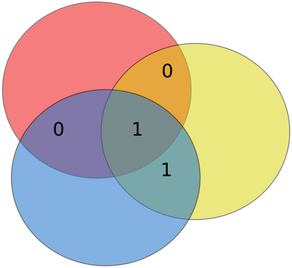


- ... then fill in the 3 circles with the sum of the bits inside that circle, taken mod 2.

Example

- If original message is [0 1 0 1], we have



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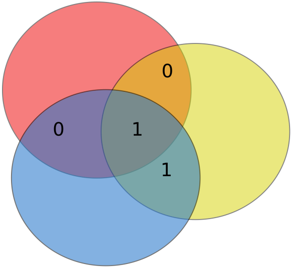

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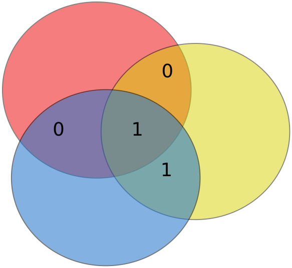

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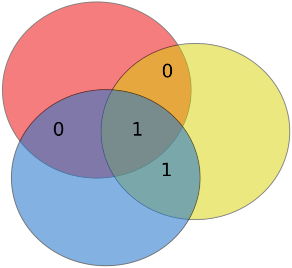
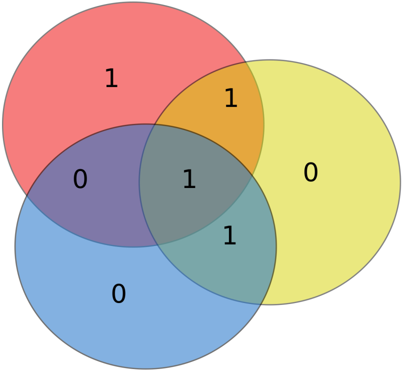


- ... filled in as?
- The sender sends 7 bits: the original four, plus the additional 3 “parity bits” (starting at top left, moving clockwise):
- [0 1 0 1 1 0 0] gets sent.

Correcting errors

- Now, recipient gets 7 bits, and fills out the picture in order. Example:



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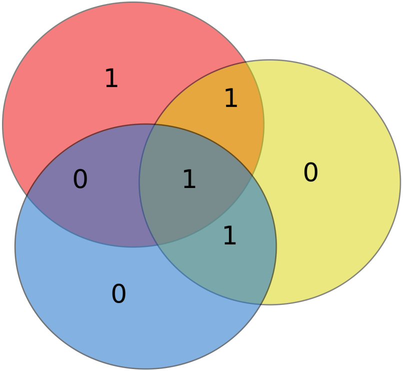

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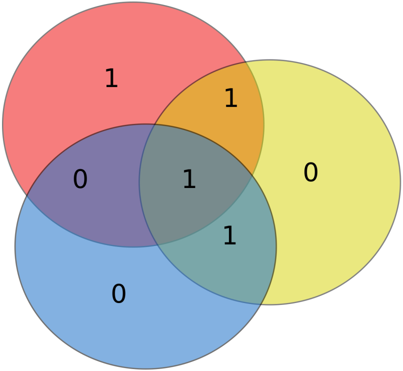

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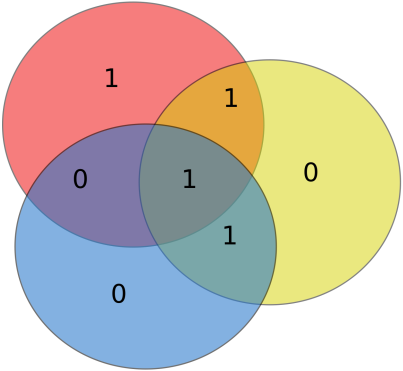

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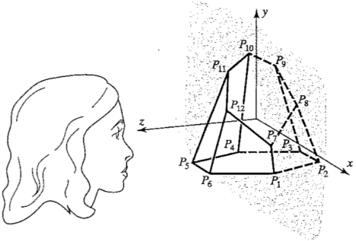
- G, P can be viewed as linear transformations.
- Messages without errors can be thought of as image of G and kernel of P. How so?

Displaying 3D graphics

- Imagine n points $(x_1, y_1, z_1), \ldots (x_n, y_n, z_n) \in R^3$, connected by various line segments:



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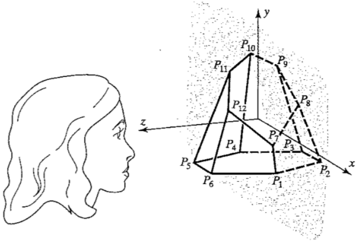

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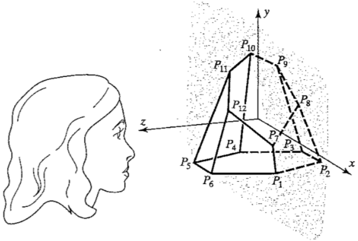

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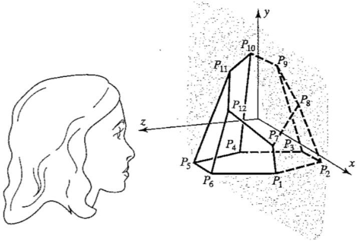

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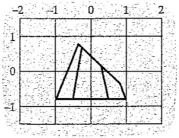
- Imagine points being projected onto a flat screen in $z = 0$ plane. Store points as a $3 \times n$ matrix:

$$P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix}$$

Views

- Straight-on view looks like



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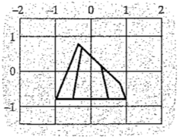

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$$S = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

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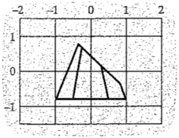

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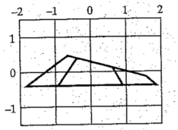
- Matrix product "SP" means apply S to each column.

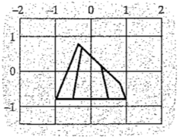
Scaling example

- Here's an example



$\alpha = 1.8, \beta = 0.5, \gamma = 3.0.$

- Original view was:



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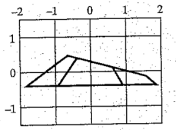

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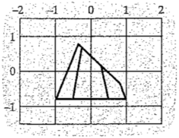

- In this initial discussion, z coordinate doesn't factor in directly.

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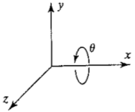
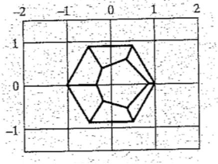


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



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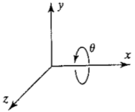

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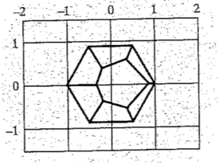
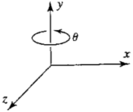


Rotation about the y-axis



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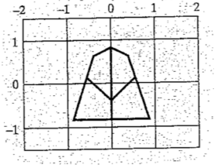

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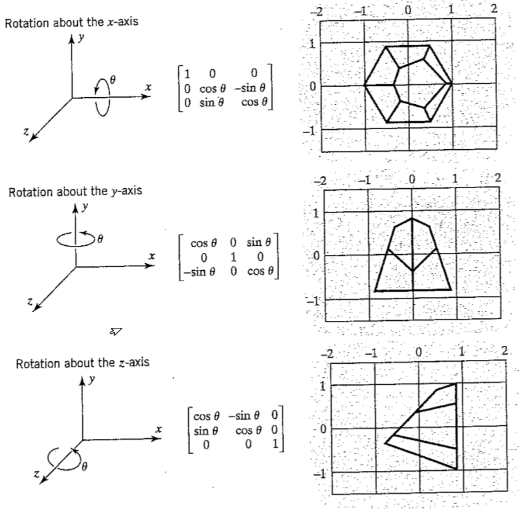

Stereoscopic views

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