Math 312, Homework 3: selected solutions

Additonal problems

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ be linear tranformations, so $S \circ T : \mathbb{R}^3 \to \mathbb{R}^3$ and $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$. Let the matrix of T be B and let the matrix of S be A.

   (a) Why must there be a vector $\vec{x} \in \mathbb{R}^3$ such that $B\vec{x} = 0$?

   (b) Prove that AB (a 3×3 matrix) can never be invertible.

   (c) Give an example to show BA (a 2×2 matrix) may be invertible.

   Proof. For (a), you can see this with "free variables", but here's a better approach. The rank of T is at most 2, since its image lives in $\mathbb{R}^2$. But the rank and nullity of T must add up to 3, so the nullity is at least 1. That is, there must be a whole line of vectors that T (or B) sends to $\vec{0}$.

   If B sends a nonzero $\vec{x}$ to zero, then so does AB. (Why?) Now, AB is a 3×3 matrix with a nontrivial kernel, so it can't be invertible. (Make sure you understand why.)

   Here's a way to find an example. Let S send (x, y) to $(x, y, 0)$, and let T send (x, y, z) to (x, y). Check the details.

   □

2. Are the following subsets of $\mathbb{R}^2$ actually subspaces? Explain.

   (a) $\{(x, y) \mid xy = 0\}$

   (b) $\{(x, y) \mid x \text{ and } y \text{ are integers}\}$

   (c) $\{(x, y) \mid x + y = 0\}$

   (d) $\{(x, y) \mid x + y \geq 0\}$

   Solution: Here are some brief explanations and hints.

   (a) is not closed under addition. Consider $\vec{e}_1, \vec{e}_2$, and $\vec{e}_1 + \vec{e}_2$.

   (b) scale a vector with integer entries by an irrational number, like $\sqrt{2}$.

   (c) This is a line through the origin, so it's a subspace. You could write it as the span of $(1, -1)$.

   (d) Consider scaling by a negative number.

3. Let T be a linear transformation with trivial kernel. Prove that if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$. (Hint: use subtraction.) Such transformations T are called *one-to-one*.

   Proof: We are told that $T(\vec{x}) = T(\vec{y})$, so that $T(\vec{x}) - T(\vec{y}) = \vec{0}$. Since T is assumed to be linear, we have $T(\vec{x} - \vec{y}) = \vec{0}$. But since T is assumed to have trivial kernel, the zero vector is the only vector that T sends to zero. This means $\vec{x} - \vec{y} = \vec{0}$, which tells us $\vec{x} = \vec{y}$.

By the way, this says that T is a "one-to-one" linear transformation.

4. Prove that if vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent, then one of these vectors is a linear combination of the others.

   Solution omitted.

5. Prove that the span of vectors $\vec{v}_1, \ldots, \vec{v}_k$ in $\mathbb{R}^m$ is always a subspace of $\mathbb{R}^m$.

   Solution omitted.

6. Say $\vec{v}_1, \ldots \vec{v}_k$ are linearly independent in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Show by example that $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ need not be linearly independent. However, prove that these vectors are linearly independent if $\ker(T)$ is trivial.

   Solution: Example: T could be project vectors in $\mathbb{R}^3$ down to a plane. Or even easier: T could send every vector to zero.

   Now, suppose $\ker(T)$ is trivial, and $\vec{v}_1, \ldots \vec{v}_k$ are linearly independent. The question is: are $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ linearly independent? Well, suppose we have scalars c_i such that

$$c_1 T(\vec{v}_1) + \ldots + c_k T(\vec{v}_k) = \vec{0}.$$

   Using the fact that T is linear tells us

$$T(c_1\vec{v}_1 + \ldots + c_k\vec{v}_k) = \vec{0}.$$

   But here we can use the fact that T has trivial kernel! The vector $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$ itself must be the zero vector. In other words, there's a linear combination of the $\vec{v}_i$'s that produces zero. But $\vec{v}_1, \ldots \vec{v}_k$ are linearly independent, so the c_i's must all be zero.

   Finally, this tells us $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ linearly independent. (Make sure you understand why.)

7. Use Theorems from section 3.3 (or from class) to explain carefully why if V and W are subspaces with V contained inside of W, then $\dim V \leq \dim W$.

   Proof. Let $\vec{v}_1, \ldots, \vec{v}_p$ be a basis of V, and let $\vec{w}_1, \ldots, \vec{w}_q$ be a basis of W. This tells us p is the dimension of V and q is the dimension of W.

   Let's use a theorem from the book or class. In our setting, we have a subspace W, with a linearly independent set $\vec{v}_1, \ldots, \vec{v}_p$ and a spanning set $\vec{w}_1, \ldots, \vec{w}_q$. (Why?) The theorem says $p \leq q$. That's it! □