Math 312, Homework 6 (due Friday, October 26th)

Name:________________________________ (if you choose to use this as a coversheet)

Reading Chapter 6 of Bretscher (and additional review of chapter 4 as needed). I also recommend you start reading chapter 7.

Book problems

- Section 6.1, problems 21, 38, 40 (use shortcuts as necessary!) (Also do as many practice determinants in 1–22 that you need to feel very comfortable with these).
- Section 6.2, problems 10, 17, 20, 38, 40 (orthogonal means $A^T A$ is the identity)
- Section 6.3, problems 2, 7, 11

Additional Problems

1. Find all solutions to the ODE $f'' - 3f' + 2f = 2x^2 - 6x + 4$ by first finding the homogeneous solutions and then also a particular solution. (We did a similar example in class.)

2. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal unit vectors in $\mathbb{R}^3$ such that $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ (cross product).

   (a) Let P be the matrix whose columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Explain why the transpose of P is also the inverse of P.

   (b) Describe a formula for the matrix, **in the standard basis**, that rotates by θ radians about the line through $\vec{v}_3$ in the direction from $\vec{v}_1$ to $\vec{v}_2$. (Hint: you know how to find this matrix in the $\vec{v}_i$ basis; you also know the change-of-basis formula.)

3. Let $V = M_{n,n}(\mathbb{R})$, the vector space of $n \times n$ matrices with real entries. Let P be an invertible $n \times n$ matrix. Define a transformation $T : V \to V$ by

   $$T(A) = PAP^{-1}.$$

   (a) Show that T is linear.

   (b) Prove that T is an isomorphism (by finding an inverse).

   (c) Interpret what the linear transformation T is doing in terms of change-of-basis.

4. Consider the problem of finding a polynomial of degree n passing through $n+1$ specified distinct points in the plane. For definiteness, take $n = 3$, and say our points are (a_1, b_1), (a_2, b_2), (a_3, b_3), and (a_4, b_4). This problem involves $\mathbb{P}_3$, and so we could work in the usual basis $\{x^3, x^2, x, 1\}$. However, we will investigate how using a different basis can make our computations simpler.

(a) Consider the cubic polynomial

$$p_1(x) = \frac{1}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)}(x - a_2)(x - a_3)(x - a_4).$$

What does $p_1(x)$ equal at the points $x = a_1, a_2, a_3, a_4$?

(b) Write down similar polynomials $p_2(x), p_3(x), p_4(x)$ that play the same roles for a_2, a_3, a_4, respectively.

(c) Prove that the set $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ is linearly independent (you won't need to multiply them out!). Since $\dim \mathbb{P}_3 = 4$, this shows we have a basis (called a Lagrange basis).

(d) What linear combination of these 4 basis polynomials produces the polynomial passing through the desired four points?

(e) Using this approach, find the polynomial of degree 3 passing through the points $(0, -3)$, $(1, -1)$, $(2, 11)$, and $(-1, -7)$.

5. Let S and T be linear transformations from a vector space V to itself. Suppose each of S and T have a 1-dimensional kernel. Our goal in this problem is to understand the dimension of the kernel of $T \circ S$.

(a) Explain why the kernel of $T \circ S$ must have dimension at least one.

(b) By finding examples using 2×2 matrices, show that it is possible for $T \circ S$ to have a one-dimensional kernel and a two-dimensional kernel (where each T and S have one-dimensional kernel).

(c) Back to the general case. Suppose there are three linearly independent vectors v_1, v_2, v_3 in the kernel of $T \circ S$. Arrive at a contradiction by showing that either T or S has a kernel of dimension 2 or greater.

Remark: This shows the dimension of the kernel of $T \circ S$ must be either 1 or 2.

6. In class we carefully showed that $f'' + f = 0$ has a 2-dimensional solution space. Here we will prove the same statement for a much larger class of ODEs. Suppose you're given the ODE:

$$f'' + bf' + cf = 0,$$

where b and c are real constants. Our usual approach is to guess a solution of the form $f(x) = e^{rx}$, which leads to the quadratic equation

$$r^2 + br + c = 0.$$

We will assume this quadratic equation has real roots r_1 and r_2, but we will NOT assume our solution has to be of the form e^{rx}.

(a) Let $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be differentiation on the vector space of infinitely differentiable functions, and let I be the identity transformation. Find the kernels of the transformations $D - r_1 I$ and $D - r_2 I$ (using single-variable calculus). They will each be one-dimensional.

(b) Prove that the kernel of $(D - r_1 I) \circ (D - r_2 I)$ is the same as the solution space of the ODE we started with.

(c) Conclude the solution space of the ODE is at most two-dimensional (using the previous problem). What is a basis for it? (consider the cases $r_1 = r_2$ and $r_1 \neq r_2$ separately).

Remark: this technique readily generalizes to nth order linear, homogeneous ODEs with constant coefficients, and also to the case in which some of the roots are complex.

7. Consider a system of four interlinked webpages, described as follows.

   - Pages 1 and 2 link to each other.
   - Pages 3 and 4 link to each other.
   - Pages 1 and 3 link to each other
   - Page 1 links to page 4.

   Find the 4×4 matrix A used in the Google PageRank algorithm for this web. Finally, by solving an appropriate linear system, determine which page gets ranked as the most important (using a computer if you wish, but it is not necessary). (Don't use the method involving 0.85 and 0.15.)

   The article `http://www.rose-hulman.edu/~bryan/googleFinalVersionFixed.pdf` explaining Google's algorithm may be helpful, but we covered enough material in class for you to solve this problem.

8. Suppose a mirror is located in the xy plane, and you are living in the region $z > 0$. What is the standard matrix of the linear transformation that sends you to your mirror image? Arguing geometrically, what is its determinant?