Fourier Series – An Example

Formulas: Let f(x) be periodic with period 2π. We want to write

$$f(x) = \frac{A_0}{2} + \sum_1^\infty \left(A_k \cos kx + B_k \sin kx\right).$$

The Fourier coefficients are given by the formulas

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \qquad B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx,$$

k=0,1,2,.... Note that we wrote the constant term in the Fourier series as $\frac{A_0}{2}$ so that the above formula for A_k also works for $k = 0$. Otherwise one needs a separate formula for A_0 incorporating the $1/2$.
Moreover one has the analogue of the "Pythagorean theorem"

$$\|f\|^2 = \pi\left[\frac{A_0^2}{2} + \sum_1^\infty (A_k^2 + B_k^2)\right].$$

Example: Consider the function

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x \le 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$$

To use the above formulas for the Fourier coefficients we split the integrals into two pieces

$$A_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos kx\, dx + \int_0^\pi (+1)\cos kx\, dx\right].$$

and

$$B_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin kx\, dx + \int_0^\pi (+1)\sin kx\, dx\right].$$

When one evaluates the A_k, the two integrals cancel so $A_k = 0$. In fact, this is easy to see immediately since our $f(x)$ is an odd function and $\cos kx$ is an even function so $f(x)\cos kx$ is an odd function. Also,

$$\int_0^\pi (+1)\sin kx\, dx = \frac{-\cos k\pi + 1}{k} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k} & \text{if } k \text{ is odd} \end{cases}$$

By a computation the first integral in B_k has the same value as this second integral. One can also see this using that $f(x)$ and $\sin kx$ are both odd functions so $f(x)\sin kx$ is an even function. Thus,

$$B_k = \frac{2}{\pi}\begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k} & \text{if } k \text{ is odd} \end{cases} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

Consequently the desired Fourier series is

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right].$$

Since our function f is an *odd* function, we should not be surprised that the right side is also an odd function. Had we used this observation, our computation would have been slightly shorter.

To use the "Pythagorean" theorem above, we observe that

$$\|f\|^2 = \int_{-\pi}^{\pi} 1^2\, dx = 2\pi.$$

Thus

$$2\pi = \frac{16}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right],$$

that is,

$$\frac{\pi^2}{8} = \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right].$$

This evaluation of the sum of the infinite series on the right is a bonus from our computation of the Fourier series of this function f.