Math 312, Fall 2012 Jerry L. Kazdan

Problem Set 1

DUE: In class Thursday, Sept. 13. *Late papers will be accepted until 1:00 PM Friday.*

These problems are intended to be straightforward with not much computation.

1. Let $S := \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

   a) Find S^{-1}.

   b) For which constant(s) λ is the matrix $\begin{pmatrix} 2-\lambda & 5 \\ 1 & 3-\lambda \end{pmatrix}$ invertible?

   c) Let $D := \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ and $B := SDS^{-1}$. Compute B^{10}.

2. Solve all of the following equations. [Note that the left sides of these equations are identical.]

   a). $\begin{aligned} 2x + 5y &= 5 \\ x + 3y &= -1 \end{aligned}$ b). $\begin{aligned} 2x + 5y &= 0 \\ x + 3y &= -2 \end{aligned}$ c). $\begin{aligned} 2x + 5y &= 1 \\ x + 3y &= 0 \end{aligned}$ d). $\begin{aligned} 2x + 5y &= 2 \\ x + 3y &= 1 \end{aligned}$

3. Let A and B be 2×2 matrices.

   a) If B is invertible and $AB = 0$, show that $A = 0$.

   b) Give and example where $AB = 0$ but $BA \neq 0$.

   c) Find an example of a 2×2 matrix with the property that $A^2 = 0$ but $A \neq 0$.

4. Consider the system of equations

$$\begin{aligned} x + y - z &= a \\ x - y + 2z &= b \\ 3x + y &= c \end{aligned}$$

   a) Find the general solution of the homogeneous equation.

   b) If $a = 1$, $b = 2$, and $c = 4$, then a particular solution of the inhomogeneous equations is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of these inhomogeneous equations.

   c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have *no* solution.

   d) If $a = 0$, $b = 0$, $c = 1$, show the equations have *no* solution. [Note: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$].

5. a) Find a real 2×2 matrix A (other than $A = \pm I$) such that $A^2 = I$.

   b) Find a real 2×2 matrix A such that $A^4 = I$ but $A^2 \neq I$.

6. Let L, M, and P be linear maps from the (two dimensional) plane to the plane:

   L is rotation by 90 degrees counterclockwise.

   M is reflection across the vertical axis

   $Nv := -v$ for any vector $v \in \mathbb{R}^2$ (reflection across the origin)

   a) Find matrices representing each of the linear maps L, M, and N.

   b) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM.

   c) Which pairs of these maps commute?

   d) Which of the following identities are correct—and why?

1) $L^2 = N$	2) $N^2 = I$	3) $L^4 = I$	4) $L^5 = L$
5) $M^2 = I$	6) $M^3 = M$	7) $MNM = N$	8) $NMN = L$

1[Last revised: January 10, 2013]