Problem Set 1

DUE: In class Thursday, Sept. 13. *Late papers will be accepted until 1:00 PM Friday.*

These problems are intended to be straightforward with not much computation.

1. Let $S := \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

   a) Find S^{-1}.

   b) For which constant(s) λ is the matrix $\begin{pmatrix} 2-\lambda & 5 \\ 1 & 3-\lambda \end{pmatrix}$ invertible?

   SOLUTION: Finding the inverse means solving the equations

   $$\begin{aligned} (2-\lambda)x_1 + 5x_2 &= y_1 \\ x_1 + (3-\lambda)x_2 &= y_2 \end{aligned}$$

   Multiply the second equation by $2-\lambda$ and subtract it from the first equation to get the equivalent system

   $$\begin{aligned} [5-(2-\lambda)(3-\lambda)]x_2 &= y_1 - (2-\lambda)y_2 \\ x_1 + (3-\lambda)x_2 &= y_2 \end{aligned}$$

   The first equation is always solvable for x_2 unless $5-(2-\lambda)(3-\lambda) = 0$. This is a quadratic equation for λ.

   c) Let $D := \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ and $B := SDS^{-1}$. Compute B^{10}.

   SOLUTION: $B^2 = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1}$, ..., $B^{10} = SD^{10}S^{-1}$. This is easy to compute since D is a diagonal matrix so $D^{10} = \begin{pmatrix} (-1)^{10} & 0 \\ 0 & 3^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3^{10} \end{pmatrix}$

2. Solve all of the following equations. [Note that the left sides of these equations are identical.]

   a). $\begin{aligned} 2x+5y &= 5 \\ x+3y &= -1 \end{aligned}$ b). $\begin{aligned} 2x+5y &= 0 \\ x+3y &= -2 \end{aligned}$ c). $\begin{aligned} 2x+5y &= 1 \\ x+3y &= 0 \end{aligned}$ d). $\begin{aligned} 2x+5y &= 2 \\ x+3y &= 1 \end{aligned}$

   SOLUTION: All of these have the form $S\vec{v} = \vec{b}$ where S is the matrix whose inverse you computed in Problem 1a). Thus $\vec{v} = S^{-1}\vec{b}$ where $\vec{b}$ is the right hand side of each of these equations. The computation is now very short.

3. Let A and B be 2×2 matrices.

   a) If B is invertible and $AB = 0$, show that $A = 0$.

      SOLUTION: Multiply the equation on the right by B^{-1} to get $A = ABB^{-1} = 0$.

   b) Give and example where $AB = 0$ but $BA \neq 0$.

      SOLUTION Let $A := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

   c) Find an example of a 2×2 matrix with the property that $A^2 = 0$ but $A \neq 0$.

      SOLUTION Let $A := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

4. Consider the system of equations

$$\begin{aligned} x + y - z &= a \\ x - y + 2z &= b \\ 3x + y &= c \end{aligned}$$

   a) Find the general solution of the homogeneous equation.

   b) If $a = 1$, $b = 2$, and $c = 4$, then a particular solution of the inhomogeneous equations is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of these inhomogeneous equations.

   c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have *no* solution.

   d) If $a = 0$, $b = 0$, $c = 1$, show the equations have *no* solution. [Note: $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \\ 4 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$].

      SOLUTION If you can solve the equation $A\vec{u} = \vec{w}$ and you *cannot* solve the equation $A\vec{v} = \vec{r}$, then you cannot solve the equation $A\vec{z} = \vec{w} - \vec{r}$ since if you could, then $\vec{v} = \vec{u} - \vec{w}$ would be a solution of $A\vec{v} = \vec{r}$.

5. a) Find a real 2×2 matrix A (other than $A = \pm I$) such that $A^2 = I$.

      SOLUTION A reflection, say across the vertical axis: $A := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$

   b) Find a real 2×2 matrix A [other than $A = \pm I$ and your answer to Part a)] such that $A^4 = I$.

      SOLUTION A rotation by 90 degrees ($\pi/2$). $A := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

6. Let L, M, and P be linear maps from the (two dimensional) plane to the plane:

   L is rotation by 90 degrees counterclockwise.

   M is reflection across the vertical axis

   $Nv := -v$ for any vector $v \in \mathbb{R}^2$ (reflection across the origin)

a) Find matrices representing each of the linear maps L, M, and N.

b) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM.

SOLUTION Note that LM is the map $LM(\vec{x}) := L(M\vec{x})$ means you first apply M to $\vec{x}$ and then apply L to the result. Thus LM means first reflect across the vertical axis and then rotate by 90 degrees counterclockwise.

c) Which pairs of these maps commute?

d) Which of the following identities are correct—and why?

1)	$L^2 = N$	2)	$N^2 = I$	3)	$L^4 = I$	4)	$L^5 = L$
5)	$M^2 = I$	6)	$M^3 = M$	7)	$MNM = N$	8)	$NMN = L$

1[Last revised: September 28, 2012]