Problem Set 2

DUE: In class Thursday, Sept. 20. *Late papers will be accepted until 1:00 PM Friday.*

Lots of problems. Most are really short.

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple mental exercises. These are *not* to be handed in.

Sec. 1.1 #32, # 35
Sec. 1.2 #2, #12
Sec. 1.3 #29, #34, #36, #37, #47, #48, #52, #53, #56, #59, #65
p. 39 #39
Sec. 2.1 #1, #2, #3, #4, #6, #19, #20, #22, #25, #32, #39, #46
Sec. 2.2 #2, #4, #8
Sec. 2.3 #4, #10, #11, #12, #13, #14, #55, #58

1. [Bretscher, Sec. 1.2 #32] Find the polynomial $f(t)$ of degree 3 such that $f(1) = 1,\ f(2) = 5,\ f'(1) = 2,\ f'(2) = 9$. Graph this polynomial.

2. [Bretscher, Sec.2.1 #13]

   a) Let $A := \left(\begin{smallmatrix} 1 & 2 \\ c & 6 \end{smallmatrix}\right)$. With your bare hands (not using anything about determinants) show that A is invertible if and only if $c \neq 3$.

      SOLUTION This asks for which c you can solve the equations

      $$\begin{aligned} x_1 + 2x_2 &= y_1 \\ cx_1 + 6x_2 &= y_2. \end{aligned}$$

      This is similar to Homework Set 1 #1b). Multiply the first equation by 3 and subtract it from the second equation. The resulting equation $(c-3)x_1 = y_2 - 3y_1$ can always be solved unless $c = 3$.

   b) Let $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. With your bare hands (not using anything about determinants) show that M is invertible if and only if $ad - bc \neq 0$. [*Hint:* Treat the cases $a \neq 0$ and $a = 0$ separately.]

      SOLUTION This is similar to the previous part – only more general.

3. [Bretscher, Sec.2.2 #6] (Review from Math 240) Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\vec{v} := (2,\,1,\,2)$ (think of thos as a column vector). Find the orthogonal projection, $P_{\mathcal{L}}\vec{z}$, of the (column) vector $\vec{z} := (1,\,1,\,1)$ onto $\mathcal{L}$.

   SOLUTION As shown in class and in the book,

   $$P_{\mathcal{L}}\vec{z} = \frac{\vec{z}\cdot\vec{v}}{\|v\|^2}\vec{v} = \frac{5}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

4. [Bretscher, Sec.2.2 #10] Let $\mathcal{L}$ be the line in $\mathbb{R}^2$ that consists of all scalar multiples of the vector $(4,3)$. Find the matrix of the orthogonal projection onto this line $\mathcal{L}$.

   SOLUTION Let $\vec{z} := (z_1,\, z_2)$ and $\vec{v} := (4,\, 3)$, where, as usual, we think of these as *column vectors*. Then as in the previous problem

$$P_{\mathcal{L}}\vec{z} = \frac{\vec{z}\cdot\vec{v}}{\|v\|^2}\vec{v} = \frac{4z_1+3z_2}{5}\begin{pmatrix}4\\3\end{pmatrix} = \frac{1}{5}\begin{pmatrix}16z_1+12z_2\\12z_1+9z_2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}16 & 12\\12 & 9\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}.$$

   Thus the matrix is $\begin{pmatrix}\frac{16}{5} & \frac{12}{5}\\ \frac{12}{5} & \frac{9}{5}\end{pmatrix}$.

5. [Bretscher, Sec.2.2 #17] Let $A := \left(\begin{smallmatrix}a & b\\ b & -a\end{smallmatrix}\right)$, where $a^2+b^2=1$. Find two perpendicular non-zero vectors $\vec{v}$ and $\vec{w}$ so that $A\vec{v}=\vec{v}$ and $A\vec{w}=-\vec{w}$ (write the entries of $\vec{v}$ and $\vec{w}$ in terms of a and b). Conclude that thinking of A as a linear map it is an orthogonal reflection across the line $\mathcal{L}$ spanned by $\vec{v}$.

   SOLUTION The equation $A\vec{v}=\vec{v}$ means $(A-I)\vec{v}=0$ s0 we solve these homogeneous equations:

$$\begin{aligned}(a-1)v_1 + bv_2 =& 0\\ bv_1 - (a+1)v_2 =& 0\end{aligned}$$

   Case 1. $b\neq 0$ so $v_1 = [(a+1)/b]\,v_2$, that is, $\vec{v} = ((a+1)/b,\, 1)v_2$.

   Case 2. $b=0$ so $a=\pm 1$. If $a=1$ then $\vec{v}=(v_1,\, 0)$. If $a=-1$, then $\vec{v}=(0,\, v_2)$

   The computation for $\vec{w}$ is similar.

6. [Bretscher, Sec.2.2 #31] Find a nonzero 3×3 matrix A so that $A\vec{x}$ is perpendicular to $\vec{v} := (1,2,3)$ for all vectors $\vec{x}\in\mathbb{R}^3$.

   SOLUTION We want $\vec{v}\cdot A\vec{x}=0$ for all x. As a computation this is straightforward: we want

$$\begin{aligned}0 = \vec{v}\cdot A\vec{x} =& 1[a_{11}x_1 + a_{12}x_2 + a_{13}x_3]\\ &+ 2[a_{21}x_1 + a_{22}x_2 + a_{23}x_3]\\ &+ 3[a_{31}x_1 + a_{32}x_2 + a_{33}x_3]\\ =&(a_{11}+2a_{21}+3a_{31})x_1\\ &+ (a_{12}+2a_{22}+3a_{32})x_2\\ &+ (a_{13}+2a_{23}+3a_{33})x_3\end{aligned}$$

   for all x_1, x_2, and x_3. This means the coefficients of x_1, x_2, and x_3 must all be zero. One way to get this is to pick a_{11}, a_{21}, and a_{31} so that $a_{11}+2a_{21}+3a_{31}=0$, say $a_{11}=2$, $a_{21}=-1$, $a_{31}=0$ and let all the other elements of A be zero.

More conceptually, if we name the three columns of A as A_1, A_2, and A_3, then notice that $A\vec{x} = A_1x_1 + A_2x_2 + A_3x_3$ (the image of A is all possible linear combinations of the columns of A). Thus we want the 3 columns of A to be orthogonal to $\vec{v}$, so we can pick A_1 to be orthogonal to $\vec{v}$ (say $a_{11} = 2$, $a_{21} = -1$, $a_{31} = 0$) and simply have $A_2 = A_3 = 0$.

7. [Bretscher, Sec.2.3 #19] Find all the matrices that commute with $A := \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$.

   SOLUTION By a straightforward computation, these matrices all have the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = aI + bA$ for any scalars a and b. [Note that for *any* square matrix M the matrices $aI + bM$ always commute with M. In this case, these are the only matrices that do so.]

8. [Bretscher, Sec.2.3 #48]

   a) Geometrically interpret the matrix $M := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$. What is M^{-1}, both geometrically and computationally?

      SOLUTION This maps (x, y) to $(x+2y, y)$ so is keeps the y coordinate unchanged but changes the x coordinate. The text (p. 64) calls this a *horizontal shear*. The inverse just changes the x coordinate in the opposite direction: $M^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. It is also a horizontal shear.

   b) If $A := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, compute AB and A^{10}.

      SOLUTION $AB = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$ so $A^{10} = \begin{pmatrix} 1 & 10a \\ 0 & 1 \end{pmatrix}$

   c) Find a 2×2 matrix C so that $C^{10} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

      SOLUTION $C = \begin{pmatrix} 1 & \frac{1}{10} \\ 0 & 1 \end{pmatrix}$

9. Which of the following sets are linear spaces?

   a) $\{X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0\}$

   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.

   c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\,dx = 0$.

   d) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$.

      SOLUTION All of these are linear spaces.

10. Proof or counterexample. Here L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

    a) If L is invertible, then L^{-1} is also invertible.

       SOLUTION The inverse of L^{-1} is just L

    b) If $L\vec{v} = 5\vec{v}$ for all vectors $\vec{v}$, then $L^{-1}\vec{w} = (1/5)\vec{w}$ for all vectors $\vec{w}$.

c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

SOLUTION True.

d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

SOLUTION True. A rotation by 315 degrees is the same as rotating by -45 degrees.

e) The zero map ($0\vec{v} := 0$ for all vectors $\vec{v}$) is invertible.

SOLUTION False. The zero map is not one-to-one since it maps all vectors to the origin.

f) The identity map ($I\vec{v} := \vec{v}$ for all vectors $\vec{v}$) is invertible.

SOLUTION True.

g) If L is invertible, then $L^{-1}0 = 0$.

SOLUTION True, since L^{-1} is a linear map and this holds for any linear map.

h) If $L\vec{v} = 0$ for some non-zero vector $\vec{v}$, then L is not invertible.

SOLUTION True since then L is not one-to-one.

i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

SOLUTION False. Reflections, say across the horizontal axes, also have this property.

11. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another.

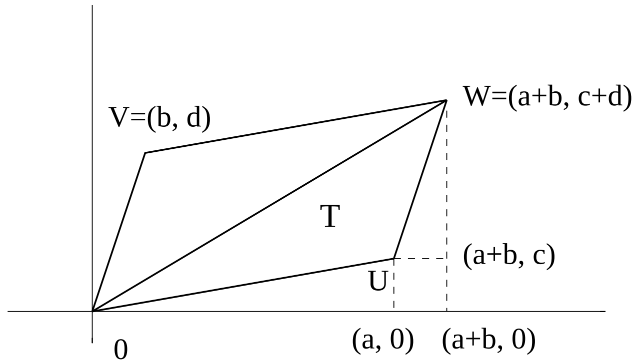
a) If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

SOLUTION For this problem it is simplest to think of a straight line as the position of a particle at time t with constant velocity $\vec{v}$, thus $\vec{x}(t) = \vec{v}t + \vec{x}_0$, where $\vec{x}_0$ is its position at $t = 0$. Then $\vec{v}$ determines the slope of the line. Another line $\vec{w}(t) = \vec{u}t + \vec{w}_0$ is parallel to this one only if $\vec{u} = c\vec{v}$ for some constant c.
The image of the straight line is $A\vec{x}(t) = (A\vec{v})t + A\vec{x}_0$. these lines all have the same vecolity vector, $A\vec{v}$ independent of the point $\vec{x}_0$ and are therefore parallel.

b) Let Q be the unit square: $0 < x < 1, 0 < y < 1$ and let Q' be its image under this map A. Give a geometric argument to show that the area$(Q') = |ad - bc|$. [More generally, the area of any region is magnified by $|ad - bc|$, which is called the *determinant* of A.]

SOLUTION The image Q' of Q is the parallelogram with vertices at the origin 0, $U := (a, c)$, $V := (b, d)$, and $W := (a+b,\ c+d)$. The triangle T with vertices at 0,

U, and W has half the area of Q'. Now consult the following figure, computing the area of the larger right triangle with vertices at 0, $(a+b,\, 0)$, and $W = (a+b,\, c+d)$



12. Let A be a matrix, not necessarily square. Say $\vec{v}$ and $\vec{w}$ are particular solutions of the equations $A\vec{v} = \vec{y}_1$ and $A\vec{w} = \vec{y}_2$, respectively, while $\vec{z} \neq 0$ is a solution of the homogeneous equation $A\vec{z} = 0$. Answer the following in terms of $\vec{v}$, $\vec{w}$, and $\vec{z}$.

    a) Find some solution of $A\vec{x} = 3\vec{y}_1$. SOLUTION: $3\vec{v}$

    b) Find some solution of $A\vec{x} = -5\vec{y}_2$. SOLUTION: $-5\vec{w}$

    c) Find some solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$. SOLUTION: $3\vec{v} - 5\vec{w}$

    d) Find another solution (other than $\vec{z}$ and 0) of the homogeneous equation $A\vec{x} = 0$.
    SOLUTION: $7\vec{z}$

    e) Find *two* solutions of $A\vec{x} = \vec{y}_1$. SOLUTION: $\vec{v}$ and $\vec{v} + 7\vec{z}$

    f) Find another solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$. SOLUTION: $3\vec{v} - 5\vec{w} + 7\vec{z}$

    g) If A is a square matrix, for any given vector $\vec{w}$ can one always find at least one solution of $A\vec{x} = \vec{w}$? Why?

    SOLUTION: No. Since the kernel of A in not just 0, A is not invertible.

13. Let V be the linear space of smooth real-valued functions and $L : V \to V$ the linear map defined by $Lu := u'' + u$.

    a) Compute $L(e^{2x})$ and $L(x)$.

    SOLUTION $L(e^{2x}) = 4e^{2x} + e^{2x} = 5e^{2x}$.
    qquad $L(x) = 0 + x = x$.

    b) Find particular solutions of the inhomogeneous equations

    $$a).\ u'' + u = 7e^{2x}, \qquad b).\ w'' + w = 4x, \qquad c).\ z'' + z = 7e^{2x} - 3x$$

    SOLUTION a). $u_{\text{part}}(x) = \frac{7}{5}e^{2x}$. b). $w_{\text{part}}(x) = 4x$. c). $z_{\text{part}}(x) = \frac{7}{5}e^{2x} - 3x$.

14. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$ and $AB : \mathbb{R}^2 \to \mathbb{R}^2$.

a) Why must there be a non-zero vector $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = 0$.

SOLUTION The equation $A\vec{x} = 0$ has 2 equations and 3 unknowns. Since there are more equations than unknowns, the homogeneous equation always has a solution other than just 0.

b) Show that the 3×3 matrix BA can *not* be invertible.

SOLUTION Let $\vec{z} \neq 0$ be some solution of $A\vec{z} = 0$. Then $(BA)\vec{z} = B(A\vec{z}) = B0 = 0$ so BA cannot be one-to-one.

c) Give an example showing that the 2×2 matrix AB might be invertible.

SOLUTION Let $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{so} \quad AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let $\mathcal{P}_N$ be the linear space of polynomials of degree at most N and $L : \mathcal{P}_N \to \mathcal{P}_N$ the linear map defined by $Lu := au'' + bu' + cu$, where a, b, and c are constants. Assume $c \neq 0$.

a) Compute $L(x^k)$.

b) Show that nullspace (=kernel) of $L : \mathcal{P}_N \to \mathcal{P}_N$ is 0.

c) Show that for every polynomial $q(x) \in \mathcal{P}_N$ there is one (and only one) solution $p(x) \in \mathcal{P}_N$ of the ODE $Lp = q$.

d) Find some solution $v(x)$ of $v'' + v = x^2 - 1$.

1[Last revised: September 28, 2012]