# Problem Set 3

DUE: In class Thursday, Sept. 27 *Late papers will be accepted until 1:00 PM Friday.*

These problems are intended to be straightforward with not much computation.

1. [BRETSCHER, SEC. 2.4 #37]. If $A$ is an invertible matrix and $c \neq 0$ is a scalar, is the matrix $cA$ invertible? If so, what is the relationship between $A^{-1}$ and $(cA)^{-1}$?

2. [BRETSCHER, SEC. 2.4 #40].
   a) If a matrix has two equal rows, show that it is not onto and thus not invertible.
   b) If a matrix has two equal columns, show that it is not one-to-one and thus not invertible.

3. [BRETSCHER, SEC. 2.4 #52]. Let $A := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix}$. Find a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, that is, it has no solution.

4. Find a real $2 \times 2$ matrix $A$ (with $A^2 \neq I$ and $A^3 \neq I$) so that $A^6 = I$. For your example, is $A^4$ invertible?

5. Let $\vec{e}_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be any non-zero vectors in $\mathbb{R}^n$.
   a) Show there is an invertible matrix $B$ with $B\vec{e}_1 = \vec{v}$.
   b) Show there is an invertible matrix $M$ with $M\vec{w} = \vec{v}$.

6. Let $A$, $B$, and $C$ be $n \times n$ matrices with $A$ and $C$ invertible. Solve the equation $ABC = I - A$ for $B$.

7. [BRETSCHER, SEC. 2.4 #67, 68, 69, 70, 71, 73] Let $A$ and $B$, be invertible $n \times n$ matrices. Which of the following are True? If False, find a counterexample.
   a) $(A+B)^2 = A^2 + 2AB + B^2$
   b) $A^2$ is invertible and $(A^2)^{-1} = (A^{-1})^2$
   c) $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$
   d) $(A-B)(A+B) = A^2 - B^2$
   e) $ABB^{-1}A^{-1} = I$
   f) $(ABA^{-1})^3 = AB^3A^{-1}$.

8. [Similar to Bretscher, Sec. 2.4 #102] Let $A$ be an $n \times n$ matrix with the property that $A^{101} = 0$.

   a) Compute $(I - A)(I + A + A^2 + \cdots + A^{100})$.

   b) Show that the matrix $I - A$ is invertible by finding its inverse.

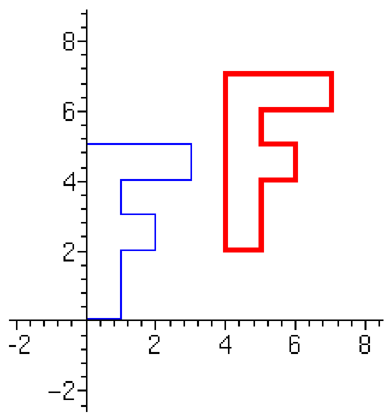
9. Find all linear maps $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is exactly the plane $\{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0 \,\}$.

10. Linear maps $F(X) = AX$, where $A$ is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

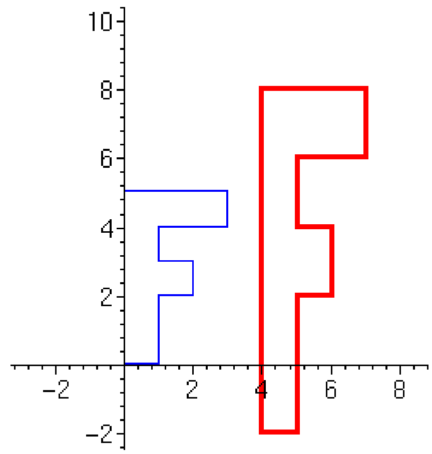
$$F(X) = V + AX,$$

where $V$ is a vector. Note that $F(0) = V$.

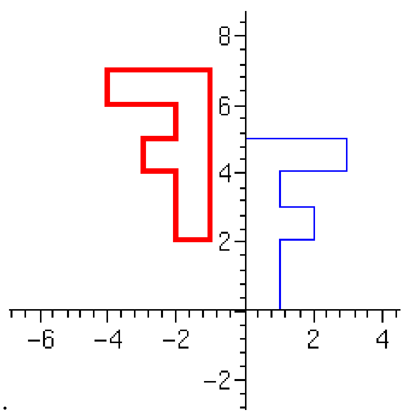
Find the vector $V$ and the matrix $A$ that describe each of the following mappings [here the light blue $F$ is mapped to the dark red $F$].



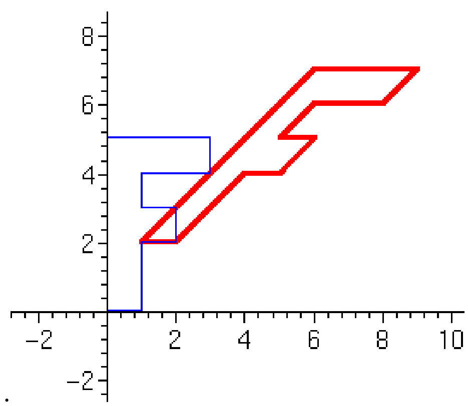

a).




b).




c).




d).

11. [Bretscher, Sec. 2.4 #35] An $n \times n$ matrix $A$ is called *upper triangular* if all the elements below the *main diagonal*, $a_{11}$ $a_{22}$, $\dots a_{nn}$ are zero, that is, if $i > j$ then $a_{ij} = 0$.

   a) Let $A$ be the upper triangular matrix

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

   For which values of $a$, $b$, $c$, $d$, $e$, $f$ is $A$ invertible?

   b) If $A$ is invertible, is its inverse also upper triangular?

   c) Show that the product of two $n \times n$ upper triangular matrices is also upper triangular.

   d) Show that an upper triangular matrix is invertible if none of the elements on the main diagonal are zero.

   e) Conversely, if an upper triangular matrix is invertible show that none of the elements on the main diagonal can be zero.

## Bonus Problem

[Please give these directly to Professor Kazdan]

1-B Let $L\colon V \to V$ be a linear map on a linear space $V$.

   a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^k \subset \ker L^{k+1}$ for all $k \ge 1$.

   b) If $\ker L^j = \ker L^{j+1}$ for some integer $j$, show that $\ker L^k = \ker L^{k+1}$ for all $k \ge j$.

   c) Let $A$ be an $n \times n$ matrix. If $A^j = 0$ for some integer $j$ (perhaps $j > n$), show that $A^n = 0$.

2-B Let $V$ be a linear space and $L : V \to V$ a linear map.

   a) Show that $\operatorname{im} L^2 \subset \operatorname{im} L$, that is, if $\vec{b}$ is in the image of $L^2$, then $\vec{b}$ is also in the image of $L$.

   b) Give and example of a $2 \times 2$ matrix $L$ where $\operatorname{im} L \not\subset \operatorname{im} L^2$.

[Last revised: February 3, 2013]