Problem Set 3

DUE: In class Thursday, Sept. 27 *Late papers will be accepted until 1:00 PM Friday.*

These problems are intended to be straightforward with not much computation.

1. [BRETSCHER, SEC. 2.4 #37]. If A is an invertible matrix and $c \neq 0$ is a scalar, is the matrix cA invertible? If so, what is the relationship between A^{-1} and $(cA)^{-1}$?

   SOLUTION $(cA)^{-1} = \dfrac{1}{c}A^{-1}$

2. [BRETSCHER, SEC. 2.4 #40].

   a) If a matrix has two equal rows, show that it is not invertible. To be more specific, it is not onto.

      SOLUTION: There are several methods, for instance, using row reduced echelon form. However, in thinking about inverses I always prefer to think in terms about solving the equation $A\vec{x} = \vec{b}$. The left side of each equation corresponds to one of the rows of A, so the left side of two of the equations are identical. Consequently there is no solution except in the rare case that their right sides of these two rows are identical. Thus A is not onto and hence not invertible.

      In greater detail, say the first two rows are identical. Then the first two equations of $A\vec{x} = \vec{b}$ are
      $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n =& b_1 \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n =& b_2 \end{aligned}.$$
      Since the left-hand sides of these are identical, these cannot have a solution unless $b_1 = b_2$. Consequently, the equation $A\vec{x} = \vec{b}$ do not have a solution except for very restricted vectors $\vec{b}$.

   b) If a matrix has two equal columns, show that it is not one-to-one and hence not invertible.

      SOLUTION: Say the columns of A are the vectors A_1, A_2,..., A_n Then the homogeneous equation $A\vec{x} = \vec{0}$ is $A_1x_1 + A_2x_2 + \cdots + A_nx_n = \vec{0}$. Say the first two columns are equal, $A_1 = A_2$. Then the homogeneous equation is $A_1(x_1 + x_2) + A_3x_3 + \cdots + A_nx_n = \vec{0}$. Clearly, any vector of the form $\vec{x} = (c, -c, 0, 0, \ldots, 0)$ is a solution for any constant c. Since the homogeneous equation has a solution other than $\vec{0}$, the kernel of A is not zero so A is not one-to-one and thus cannot be invertible.

   REMARK: Neither part of this used that A is a square matrix.

3. [BRETSCHER, SEC. 2.4 #52]. Let $A := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix}$. Find a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, that is, it has no solution.

Solution: Call the columns A_1, A_2, A_3, so the equation $A\vec{x} = \vec{b}$ is $A_1x_1 + A_2x_2 + A_3x_3 = \vec{b}$. Note that in this problem $A_3 = 2A_2$. Thus the equation is $A_1x_1 + A_2(x_2 + 2x_3) = \vec{b}$ so $\vec{b}$ must be a linear combination of A_1 and A_2. A simple $\vec{b}$ *not* of this form is $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

4. Find a real 2×2 matrix A (with $A^2 \neq I$ and $A^3 \neq I$) so that $A^6 = I$. For your example, is A^4 invertible?

   Solution: Let A be a rotation by 60 degrees. A^4 is invertible since $A^4A^2 = A^6 = I$. Thus $(A^4)^{-1} = A^2$. Alternatively, for this example, since this A is a rotation (and any rotation is invertible), this is geometrically obvious.

5. Let $\vec{e}_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be any non-zero vectors in $\mathbb{R}^n$.

   a) Show there is an invertible matrix B with $B\vec{e}_1 = \vec{v}$.

      Solution: Let the first column of B be the vector $\vec{v}$ and for the remaining columns use any vectors that extend $\vec{v}$ to a basis for $\mathbb{R}^n$. For instance, if the first component of $\vec{v}$ is not zero, you can use the standard basis vectors $\vec{e}_2, \ldots, \vec{e}_n$ for the remaining columns of B.

   b) Show there is an invertible matrix M with $M\vec{w} = \vec{v}$.

      Solution: As in the previous part, let A be an invertible matrix that maps $\vec{e}_1$ to $\vec{w}$. Then let $M := BA^{-1}$.

6. Let A, B, and C be $n \times n$ matrices with A and C invertible. Solve the equation $ABC = I - A$ for B.

   Solution: $B = A^{-1}(I - A)C^{-1}$. You can rewrite this in various ways – but I won't. However, one must be careful since the matrices A, B, and C are not assumed to commute.

7. [Bretscher, Sec. 2.4 #67, 68, 69, 70, 71, 73] Let A and B, be invertible $n \times n$ matrices. Which of the following are True? If False, find a counterexample.

   a) $(A + B)^2 = A^2 + 2AB + B^2$

      Solution: False – unless $AB = BA$. Correct version: $(A + B)^2 = A^2 + AB + BA + B^2$

   b) A^2 is invertible and $(A^2)^{-1} = (A^{-1})^2$

      Solution: True

c) $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$

SOLUTION: False, for instance if $B = -A$,

d) $(A-B)(A+B) = A^2 - B^2$

SOLUTION: False unless $AB = BA$. Should have $(A-B)(A+B) = A^2 + AB - BA - B^2$

e) $ABB^{-1}A^{-1} = I$

SOLUTION: True since $BB^{-1} = I$.

f) $(ABA^{-1})^3 = AB^3A^{-1}$.

SOLUTION: True since $(ABA^{-1})^3 = (ABA^{-1})(ABA^{-1})(ABA^{-1}) = AB^3A^{-1}$

8. [SIMILAR TO BRETSCHER, SEC. 2.4 #102] Let A be an $n \times n$ matrix with the property that $A^{101} = 0$.

a) Compute $(I-A)(I + A + A^2 + \cdots + A^{100})$.

SOLUTION: In the special case where $A^4 = 0$ this is $(I-A)(I+A+A^2+A^3) = ((I+A+A^2+A^3) - (A+A^2+A^3+A^4)) = I - A^4 = I$.
The case where $A^{101} = 0$ is essentially identical: $(I-A)(I+A+A^2+\cdots+A^{100}) = I$

b) Show that the matrix $I - A$ is invertible by finding its inverse.

SOLUTION: From the previous part, $(I-A)^{-1} = I + A + A^2 + \cdots + A^{100}$.

9. Find all linear maps $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is exactly the plane $\{\,(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\,\}$.

SOLUTION: The first step is to get coordinates for the points $\vec{x} = (x_1,\, x_2,\, x_3)$ in this plane. Solve its equation for, say, x_3, so $x_3 = x_1 + 2x_2$. Since we want $A\vec{x} = 0$, we write $\vec{x}$ as a column vector and then use x_1 and x_2 as coordinates:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + 2x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = x_1\vec{v} + x_2\vec{w},$$

where $\vec{v}_1$ and $\vec{v}_2$ are the two obvious column vectors. Since we want $A\vec{x} = 0$ for the vectors in the plane, this means

$$0 = A(x_1\vec{v} + x_2\vec{w}) = x_1A\vec{v} + x_2A\vec{w}.$$

Because x_1 and x_2 can be anything, we want $A\vec{v} = 0$ and $A\vec{w} = 0$, that is,

$$0 = A\vec{v} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{13} \\ a_{21} + a_{23} \\ a_{31} + a_{33} \end{pmatrix} = A_1 + A_3,$$

where A_1, A_2, and A_3 are the three columns of A. This gives $A_1 = -A_3$.
Similarly, since we want $A\vec{w} = 0$,

$$0 = A\vec{w} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a_{12} + 2a_{13} \\ a_{22} + 2a_{23} \\ a_{32} + 2a_{33} \end{pmatrix} = A_2 + 2A_3,$$

so $A_2 + 2A_3 = 0$, that is, $A_2 = -2A_3$.
To summarize, we find that

$$A = \left(\left(-A_3 \right) \left(-2A_3 \right) \left(A_3 \right) \right),$$

where A_3 can be any column vector, say $A_3 = (a, b, c)$, then

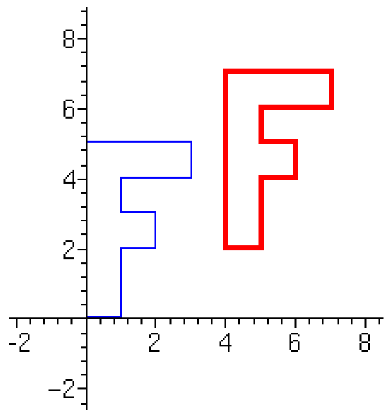
$$A = \begin{pmatrix} -a & -2a & a \\ -b & -2b & b \\ -c & -2c & c \end{pmatrix}$$

10. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

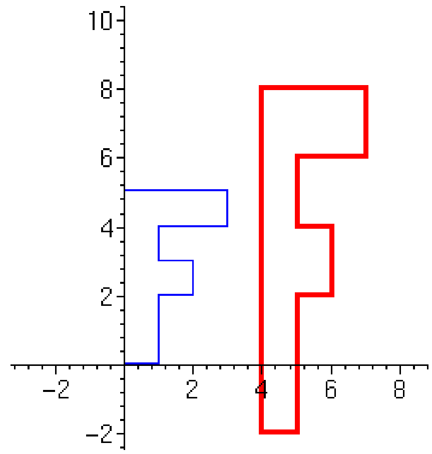
$$F(X) = V + AX,$$

where V is a vector. Note that $F(0) = V$.

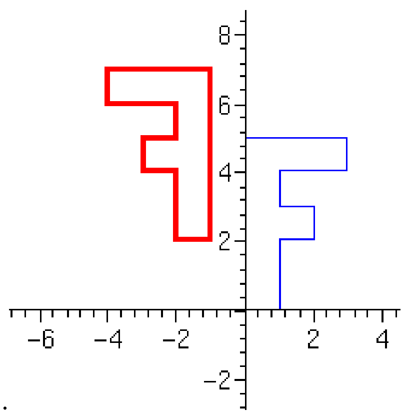
Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].



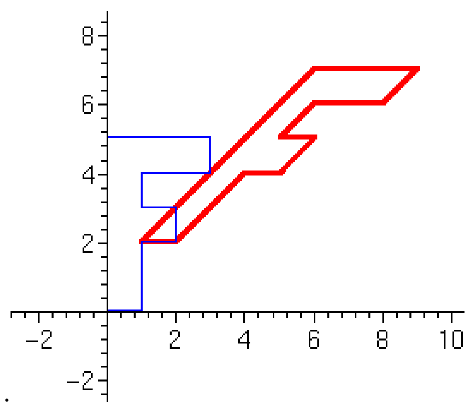
a).



b).



c).



d).

SOLUTION:

a). $V = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad A = I$ b). $V = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

c). $V = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ d). $V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

11. [Bretscher, Sec. 2.4 #35] An $n \times n$ matrix A is called *upper triangular* if all the elements below the *main diagonal*, a_{11}, a_{22}, $\ldots a_{nn}$ are zero, that is, if $i > j$ then $a_{ij} = 0$.

a) Let A be the upper triangular matrix

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

For which values of a, b, c, d, e, f is A invertible?

Solution: As always, in thinking about the invertability I think of solving the equations $A\vec{x} = \vec{y}$. In this case, the equations are

$$\begin{aligned} ax_1 + bx_2 + cx_3 =& y_1 \\ + dx_2 + ex_3 =& y_2 \\ fx_3 =& y_3 \end{aligned}$$

Clearly, to always be able to solve the last equation for x_3 we need $f \neq 0$. This gives us x_3, which we use in the second equation. It then can always be solved for x_2 if (and only if) $d \neq 0$. Inserting the values of x_2 and x_3 in the first equation, it can always be solved for x_1 if (and only if) $a \neq 0$.

Summary: An upper triangular matrix A is invertible if and only if none of its diagonal elements are 0.

b) If A is invertible, is its inverse also upper triangular?

Solution: In the above computation, notice that x_3 only depends on y_3. Then x_2 only depends on y_2 and y_3. Finally, x_1 depends on y_1, y_2, and y_3. Thus the inverse matrix is also upper triangular.

c) Show that the product of two $n \times n$ upper triangular matrices is also upper triangular.

Solution: Try the 3×3 case.

The general case is the same – but takes some thought to write-out clearly and briefly. It is a consequence of three observations:

1. A matrix $C := \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$ is upper-triangular if all the elements below the main diagonal are zero, that is, $c_{jk} = 0$ for all $j > k$.

2. For any matrices, to compute the product AB, the jk element is the dot product of the j^{th} row of A with the k^{th} column of B.

3. For upper-triangular matrices:

the j^{th} row of A is $(0, \ldots 0, a_{jj}, \ldots, a_{jn})$ while the k^{th} column of B is $\begin{pmatrix} b_{1k} \\ \vdots \\ b_{kk} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

For $j > k$, take the dot product of these vectors. The result is now obvious.

d) Show that an upper triangular matrix is invertible if none of the elements a_{11}, a_{22}, a_{33}, ..., a_{nn}, on the main diagonal are zero.

SOLUTION: This is the same as part a). The equations $A\vec{x} = \vec{y}$ are

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1\,n-1}x_{n-1} + \quad a_{1n}x_n &= y_1 \\
a_{22}x_2 + \cdots + a_{2\,n-1}x_{n-1} + \quad a_{2n}x_n &= y_2 \\
\vdots \qquad \vdots \qquad\qquad & \vdots \\
a_{n-1\,n-1}x_{n-1} + \quad a_{n-1\,n}x_n &= y_{n-1} \\
a_{nn}x_n &= y_n.
\end{aligned}$$

To begin, solve the last equation for x_n. This can always be done if (and only if) $a_{nn} \neq 0$. Then solve the second from the last for x_{n-1}, etc. This computation also proves the converse (below).

As in part b), the inverse, if it exists, is also upper triangular.

e) Conversely, if an upper triangular matrix is invertible show that none of the elements on the main diagonal can be zero.

Bonus Problem

[Please give these directly to Professor Kazdan]

1-B Let $L\colon V \to V$ be a linear map on a linear space V.

a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^k \subset \ker L^{k+1}$ for all $k \geq 1$.

b) If $\ker L^j = \ker L^{j+1}$ for some integer j, show that $\ker L^k = \ker L^{k+1}$ for all $k \geq j$.

c) Let A be an $n \times n$ matrix. If $A^j = 0$ for some integer j (perhaps $j > n$), show that $A^n = 0$.

2-B Let V be a linear space and $L : V \to V$ a linear map.

a) Show that $\operatorname{im} L^2 \subset \operatorname{im} L$, that is, if $\vec{b}$ is in the image of L^2, then $\vec{b}$ is also in the image of L.

b) Give and example of a 2×2 matrix L where $\operatorname{im} L \not\subset \operatorname{im} L^2$.

[Last revised: February 7, 2013]