Problem Set 4

DUE: In class Thursday, Oct. 4 *Late papers will be accepted until 1:00 PM Friday.*

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple mental exercises. These are *not* to be handed in.

Sec. 3.1, #2, 4, 13, 15, 30
Sec. 3.2 #1, 2, 9. 10, 11, 19, 27
Sec. 3.3 #1, 2, 28, 32, 35, 37, 39
Sec. 3.4 #1, 2, 3, 10, 20, 37
Sec. 4.1 # 1-5, 8, 10, 25, 26, 48, 50

1. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

   a). $\{(0,\ 1),\ (1,\ 1)\}$
   b). $\{(1,\ 0),\ (0,\ 1),\ (1,\ 1)\}$
   c). $\{(1,\ 0),\ (-1,\ 0\}$
   d). $\{(1,\ 1),\ (1,\ -1)\}$
   e). $\{((1,\ 1),\ (2,\ 2)\}$
   f). $\{(1,\ 2)\}$

2. For which real numbers x do the vectors: $(x,1,1)$, $(1,x,1)$, $(1,1,x)$, *not* form a basis of $\mathbb{R}^3$? For each of the values of x that you find, what is the dimension of the subspace of $\mathbb{R}^3$ that they span?

3. Compute the dimension and find bases for the following linear spaces.
   a) [SEE BRETSCHER, SEC. 4.1 #25]. Quartic polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ with the property that $p(2) = 0$ and $p(3) = 0$.
   b) [SEE BRETSCHER, SEC. 4.1 #8] Real upper triangular 3×3 matrices (first, show that this is a linear space).
   c) The space of linear maps $L : \mathbb{R}^5 \to \mathbb{R}^3$ whose kernels contain $(0, 2, -3, 0, 1)$.

4. [SEE BRETSCHER, SEC. 3.2 #6] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and let $W = U \cap V$. Find all possible values for the dimension of W.

5. [SEE BRETSCHER, SEC. 3.2 #50] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.
   a) Show that W is a linear space.
   b) Find all possible values for the dimension of W.

6. [SEE BRETSCHER, SEC. 3.2 #42] Let $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ be orthogonal unit vectors in $\mathbb{R}^n$. Show that they must be linearly independent. [This problem s very short.]

7. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

   b) If $n > k$ you can *always* solve $AX = Y$.

   c) If $n > k$ the nullspace of A has dimension greater than zero.

   d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

   e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

8. a) Find a 3×3 matrix that acts on $\mathbb{R}^3$ as follows: it keeps the x_1 axis fixed but rotates the x_2 x_3 plane by 60 degrees.

   b) Find a 3×3 matrix A mapping $\mathbb{R}^3 \to \mathbb{R}^3$ that rotates the x_1 x_3 plane by 60 degrees and leaves the x_2 axis fixed.

9. Give a proof or counterexample the following. In each case your answers should be brief.

   a) Suppose that u, v and w are vectors in a vector space V and $T : V \to W$ is a linear map. If u, v and w are linearly dependent, is it true that $T(u)$, $T(v)$ and $T(w)$ are linearly dependent? Why?

   b) If $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map, is it possible that the nullspace of T is one dimensional?

10. Find a polynomial $p(x)$ of degree at most 3 that passes through the following 4 points in the plane $\mathbb{R}^2$: $(1, 1)$, $(2, 0)$, $(3, -1)$, and $(4, 3)$.

11. [Bretscher, Sec. 3.1 #37] For the matrix $M := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ describe the kernels and images of A, A^2, and A^3 geometrically.

12. [Bretscher, Sec. 3.2 #46] Find a basis for the kernel of the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{pmatrix}.$$

Justify your answer carefully; that is, explain how the vectors you found are both linearly independent and span the kernel.

13. Compute the rank (dimension of the image) of each of the following matrices.

$a).\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $b).\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ $c).\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 7 \\ 0 & 0 & 2 \end{pmatrix}$ $d).\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ $e).\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{pmatrix}$

14. Compute the rank (dimension of the image) of each of the following matrices.

$a).\ \begin{pmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -5 \end{pmatrix}$ $b).\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $c).\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

15. [Bretscher, Sec. 3.3 #30] Find a basis for the subspace of $\mathbb{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

16. [Bretscher, Sec. 4.1 #51] Find all solutions $f(x)$ of the differential equation

$$f'' - 7f' + 12f = 0.$$

17. [Remark: Bretscher, Sec. 4.1 #58] was done in class. This is not a homework problem.

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B [Bretscher, Sec. 3.3 #64] Two subspaces V and W of $\mathbb{R}^n$ are called *complements* if any vector $\vec{x} \in \mathbb{R}^n$ can be expressed uniquely as $\vec{x} = \vec{v} + \vec{w}$, where $\vec{v} \in V$ and $\vec{w} \in W$. Show that V and W are complements if (and only if) $V \cap W = 0$ and $\dim V + \dim W = n$.

[Last revised: December 22, 2012]