Math 312, Fall 2012 Jerry L. Kazdan

Problem Set 5

DUE: In class Thursday, Oct. 18 *Late papers will be accepted until 1:00 PM Friday.*

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple mental exercises. These are *not* to be handed in.

Sec. 5.1, #28, 29, 31
Sec. 5.2 #33

1. [BRETSCHER, SEC. 5.1 #16] Consider the following vectors in $\mathbb{R}^4$

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \qquad \vec{u}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

Can you find a vector u_4 in $\mathbb{R}^4$ such that the vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, $\vec{u}_4$ are orthonormal? If so, how many such vectors are there?

2. [BRETSCHER, SEC. 5.1 #17] Find a basis for $W^{\perp}$, where

$$W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right\}.$$

3. [BRETSCHER, SEC. 5.1 #21] Find scalars a, b, c, d, e, f, and g so that the following vectors are orthonormal:

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \qquad \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \qquad \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}.$$

4. [BRETSCHER, SEC. 5.1 #26] Find the orthogonal projection P_S of $\vec{x} := \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix}$ into the subspace S of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

5. [BRETSCHER, SEC. 5.1 #37] Consider a plane V in $\mathbb{R}^3$ with orthonormal basis $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x}$ be a vector in $\mathbb{R}^3$. Find a formula for the reflection $R\vec{x}$ of $\vec{x}$ across the plane V.

6. [Bretscher, Sec. 5.2 #32] Find an orthonormal basis for the plane $x_1+x_2+x_3=0$.

7. Let V be a linear space. A linear map $P\colon V \to V$ is called a *projection* if $P^2 = P$ (this P is not necessarily an "orthogonal projection").

   a) Show that the matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a projection. Draw a sketch of $\mathbb{R}^2$ showing the vectors $(1,2)$, $(-1,0)$, $(3,1)$,and $(0,3)$ and their images under the map P. Also indicate both the image, V, and kernel, W, of P.

   b) Repeat this for the complementary projection $Q := I - P$.

   c) If the image and kernel of a projection P are orthogonal then P is called an *orthogonal projection.* [This of course now assumes that V has an inner product.] Let $M = \left(\begin{smallmatrix} 0 & a \\ 0 & c \end{smallmatrix}\right)$. For which real value(s) of a and c is this a projection? An orthogonal projection?

The remaining problems are from the Lecture notes on Vectors

`http://www.math.upenn.edu/~kazdan/312F12/notes/vectors/vectors8.pdf`

8. [p. 8 #5] The origin and the vectors X, Y, and $X+Y$ define a parallelogram whose diagonals have length $X+Y$ and $X-Y$. Prove the *parallelogram law*

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2;$$

   This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

9. [p. 8 #6]

   a) Find the distance from the straight line $3x - 4y = 10$ to the origin.

   b) Find the distance from the plane $ax + by + cz = d$ to the origin (assume the vector $\vec{N} = (a,b,c) \neq 0$).

10. [p. 8 #8]

    a) If X and Y are real vectors, show that

$$\langle X,\, Y\rangle = \frac{1}{4}\left(\|X+Y\|^2 - \|X-Y\|^2\right).$$

    This formula is the simplest way to recover properties of the inner product from the norm.

    b) As an application, show that if a square matrix R has the property that it preserves length, so $\|RX\| = \|X\|$ for every vector X, then it preserves the inner product, that is, $\langle RX,\, RY\rangle = \langle X,\, Y\rangle$ for all vectors X and Y.

11. [p. 9 #10]

    a) If a certain matrix C satisfies $\langle X, CY\rangle = 0$ for *all* vectors X and Y, show that $C = 0$.

    b) If the matrices A and B satisfy $\langle X, AY\rangle = \langle X, BY\rangle$ for all vectors X and Y, show that $A = B$.

12. [p. 9 #11–12] A matrix A is called *anti-symmetric* (or skew-symmetric) if $A^* = -A$.

    a) Give an example of a 3×3 anti-symmetric matrix.

    b) If A is any anti-symmetric matrix, show that $\langle X, AX\rangle = 0$ for all vectors X.

    c) Say $X(t)$ is a solution of the differential equation $\dfrac{dX}{dt} = AX$, where A is an anti-symmetric matrix. Show that $\|X(t)\| = \text{constant}$. [REMARK: A special case is that $X(t) := \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$ satisfies $X' = AX$ with $A = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ so this problem gives another proof that $\cos^2 t + \sin^2 t = 1$.]

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B This is a followup to problem 7.

   a) If a projection P is self-adjoint, so $P^* = P$, show that P is an orthogonal projection.

   b) Conversely, if P is an orthogonal projection, show that it is self-adjoint.

[Last revised: October 22, 2012]