## Problem Set 5

DUE: In class Thursday, Oct. 18 *Late papers will be accepted until 1:00 PM Friday.*

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple mental exercises. These are *not* to be handed in.

Sec. 5.1, #28, 29, 31
Sec. 5.2 #33

1. [BRETSCHER, SEC. 5.1 #16] Consider the following vectors in $\mathbb{R}^4$

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \qquad \vec{u}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

Can you find a vector $u_4$ in $\mathbb{R}^4$ such that the vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, $\vec{u}_4$ are orthonormal? If so, how many such vectors are there?

SOLUTION: Since $\mathbb{R}^4$ is four dimensional, you can extend these three orthonormal vectors to an orthonormal basis. First find a vector orthogonal to these three; then normalize it to be a unit vector. In this case, looking at the three given vectors, another immediately comes to mind:

$$\vec{u}_4 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \tag{1}$$

with $\hat{\vec{u}} := -\vec{u}_4$ another possibility. These two are the only possibility since the orthogonal complement of the span of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ is one dimensional so a basis will have only one vector. After we have found one, which we call $\vec{u}_4$, any other, say $\hat{\vec{w}}$ must have the form $\vec{w} = c\vec{u}_4$ for some constant $c$. Because we want a unit vector,

$$1 = \|\vec{w}\| = c^2\|\vec{u}_4\| = c^2,$$

so $c = \pm 1$.

But what if my $\vec{u}_4$ didn't immediately come to mind? Use the Gram-Schmidt process. Pick any vector *not* in the span of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$. Almost any vector in $\mathbb{R}^4$ will do. I will try the simple $\vec{w} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. We want to write $\vec{w}$ in the form

$$\vec{w} = a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + \vec{z},$$

where $\vec{z}$ is orthogonal to $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$. Taking the inner product of both sides of this successively with $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ (which are unit vectors), we find that

$$a = \langle \vec{w}, \vec{u}_1 \rangle = 1/2, \qquad b = \langle \vec{w}, \vec{u}_2 \rangle = 1/2, \qquad c = \langle \vec{w}, \vec{u}_3 \rangle = 1/2.$$

Then

$$\vec{z} = \vec{w} - [(1/2)\vec{u}_1 + (1/2)\vec{u}_2 + (1/2)\vec{u}_3] = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{pmatrix}$$

To get the desired unit vector we let $\vec{u}_4 = \vec{z}/\|\vec{z}\|$ which agrees with (1)

2. [Bretscher, Sec. 5.1 #17] Find a basis for $W^\perp$, where

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right\}.$$

Solution: The vectors $\vec{x} = (x_1, x_2, x_3, x_4) \in W^\perp$ must satisfy

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= 0 \end{aligned}$$

Solving these equations for $x_1$ and $x_2$ in terms of $x_3$ and $x_4$ we find

$$x_1 = x_3 + 2x_4 \qquad x_2 = -2x_3 - 3x_4.$$

Thus

$$\vec{x} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} x_4$$

The two vectors at the end of the previous line are a basis for $W^\perp$

3. [Bretscher, Sec. 5.1 #21] Find scalars $a$, $b$, $c$, $d$, $e$, $f$, and $g$ so that the following vectors are orthonormal:

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \qquad \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \qquad \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}.$$

Solution: The orthogonality gives

$$ab + d + fg = 0, \qquad ac + ed + f/2 = 0, \qquad bc + e + g/2 = 0.$$

Because we want unit vectors, so we can't scale the second or third vectors, we need $b = g = 0$ and we can't simply let $c = e = 0$ (it took me a few minutes to grasp this). The orthogonality conditions are then

$$d = 0, \qquad ac + f/2 = 0, \qquad e = 0.$$

That these are unit vectors gives $a^2 + f^2 = 1$ and $c^2 + 1/4 = 1$. Therefore $c = \pm\sqrt{3}/2$, so $f = \mp(\sqrt{3})a$, which in turn implies $a = \pm 1/2$.

4. [Bretscher, Sec. 5.1 #26] Find the orthogonal projection $P_S$ of $\vec{x} := \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix}$ into the subspace $S$ of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

Solution: We are fortunate that the vectors $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. We want to find constants $a$ and $b$ so that

$$\vec{x} = a\vec{v}_1 + b\vec{v}_2 + \vec{w}, \tag{2}$$

where $\vec{w}$ is orthogonal to $S$. Then the desired projection will be $P_S\vec{x} = a\vec{v}_1 + b\vec{v}_2$. To find the scalars $a$ and $n$, take the inner product of (2) with $\vec{v}_1$ and then $\vec{v}_2$ we find

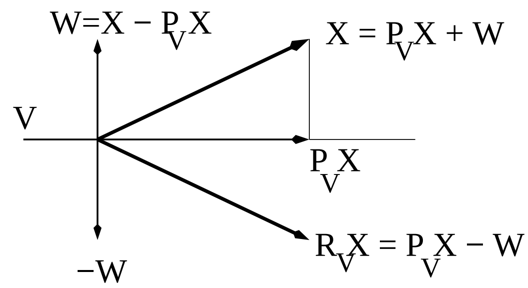
$$\langle \vec{x}, \vec{v}_1 \rangle = a\|\vec{v}_1\|^2 \qquad \text{and} \qquad \langle \vec{x}, \vec{v}_2 \rangle = a\|\vec{v}_2\|^2.$$

Using the particular vectors in this problem, $a = 11$ and $b = -1$. Thus

$$P_S\vec{x} = 11\vec{v}_1 - \vec{v}_2 = \begin{pmatrix} 19 \\ 39 \\ 64 \end{pmatrix}$$

5. [Bretscher, Sec. 5.1 #37] Consider a plane $V$ in $\mathbb{R}^3$ with orthonormal basis $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x}$ be a vector in $\mathbb{R}^3$. Find a formula for the reflection $R\vec{x}$ of $\vec{x}$ across the plane $V$.

Solution: The key is a picture (first try it in $\mathbb{R}^2$ where $V$ is a line through the origin). Let $P_V\vec{x}$ be the orthogonal projection of $\vec{x}$ into the plane $V$. Then $\vec{w} := P_{V^\perp}\vec{x} = \vec{x} - P_V\vec{x}$ is the projection of $\vec{x}$ orthogonal to $V$. From the picture, to get the reflection, replace $\vec{w}$ by $-\vec{w}$




Thus, since $\vec{x} = P_V\vec{x} + \vec{w}$, then

$$R_V\vec{x} = P_V\vec{x} - \vec{w} = P_V\vec{x} - (\vec{x} - P_V\vec{x}) = 2P_V\vec{x} - \vec{x}.$$

In summary, orthogonal projections and reflections for a subspace $V$ are related by the simple formula $R_V = 2P_V - I$.

Note that the orthogonal projection, $P_V\vec{x}$, is easy to compute if you know an orthonormal basis. All of this is very general. In this problem $\vec{u}_1$ and $\vec{u}_2$ are an orthonormal basis for the subspace $V$, so

$$P_V\vec{x} = \langle\vec{x},\,\vec{u}_1\rangle\vec{u}_1 + \langle\vec{x},\,\vec{u}_2\rangle\vec{u}_2.$$

Consequently,

$$R_V\vec{x} = 2(\langle\vec{x},\,\vec{u}_1\rangle\vec{u}_1 + \langle\vec{x},\,\vec{u}_2\rangle\vec{u}_2) - \vec{x}.$$

6. [Bretscher, Sec. 5.2 #32] Find an orthonormal basis for the plane $x_1+x_2+x_3=0$.

Solution: Pick any point in the plane, say $\vec{v}_1 = (1,-1,0)$. This will be the first vector in our orthogonal basis. We use the Gram-Schmidt process to extend this to an orthogonal basis for the plane.

Pick any other point in the plane, say $\vec{w}_1 := (1,0-1)$. Write it as $\vec{w}_1 = a\vec{v}_1 + \vec{z}$, where $\vec{z}$ is perpendicular to $\vec{v}_1$. Note that, although unknown, $\vec{z}$ will also be in the plane since it will be a linear combination of $\vec{v}_1$ and $\vec{w}$, both of which are in the plane. As usual, by taking the inner product of both sides of $\vec{w}_1 = a\vec{v}_1 + \vec{z}$ with $\vec{v}_1$, we find

$$a = \langle\vec{w}_1,\,\vec{v}_1\rangle/\|\vec{v}_1\|^2 = \frac{1}{2}.$$

Thus

$$\vec{z} = \vec{w}_1 - \frac{1}{2}\vec{v}_1 = \begin{pmatrix}1/2\\1/2\\-1\end{pmatrix}$$

is in the plane and orthogonal to $\vec{v}_1$. The vectors $\vec{v}_1$ and $\vec{z}$ are an orthogonal basis for this plane. To get an *orthonormal* basis we just make these into unit vectors

$$\vec{u}_1 := \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} \qquad\text{and}\qquad \vec{u}_2 := \frac{\vec{z}}{\|\vec{z}\|} = \frac{1}{\sqrt{3/2}}\begin{pmatrix}1/2\\1/2\\-1\end{pmatrix}$$

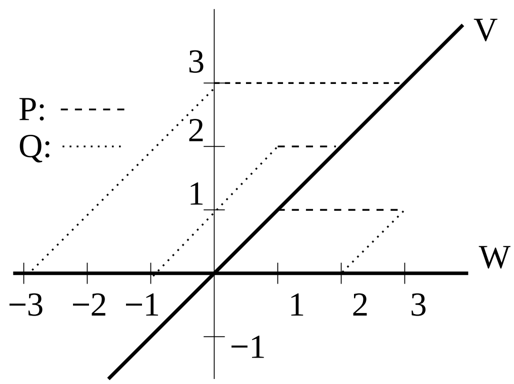
7. Let $V$ be a linear space. A linear map $P\colon V \to V$ is called a *projection* if $P^2 = P$ (this $P$ is not necessarily an "orthogonal projection").

a) Show that the matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a projection. Draw a sketch of $\mathbb{R}^2$ showing the vectors $(1,2)$, $(-1,0)$, $(3,1)$ and $(0,3)$ and their images under the map $P$. Also indicate both the image, $V$, and kernel, $W$, of $P$.

SOLUTION: The image of $P$ is the line $x_1 = x_2$ (the subspace $V$); the kernel is the $x_1$ axis (the subspace $W$). See the figure below.

b) Repeat this for the complementary projection $Q := I - P$.

SOLUTION: The image of $Q$ is the $x_1$ axis (the subspace $W$); its kernel is the line $x_1 = x_2$ (the subspace $V$).




c) If the image and kernel of a projection $P$ are orthogonal then $P$ is called an *orthogonal projection.* [This of course now assumes that $V$ has an inner product.] Let $M = \left(\begin{smallmatrix} 0 & a \\ 0 & c \end{smallmatrix}\right)$. For which real value(s) of $a$ and $c$ is this a projection? An orthogonal projection?

SOLUTION: $M^2 = \begin{pmatrix} 0 & ac \\ 0 & c^2 \end{pmatrix}$ so $M^2 = M$ requires that $ac = a$ and $c^2 = c$. The first requires that either $a = 0$ or $c = 1$. If $a = 0$ the second equation is satisfied if either $c = 0$ or $c = 1$. If $a \neq 0$, then $c = 1$. Thus, the possibilities are:

$$P_1 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{or} \qquad P_2 := \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} \quad \text{(for any } a\text{)}.$$

For an orthogonal projection $P$, its image and kernal must be orthogonal. Since for $P_1$ its image is just $0$, which is orthogonal to everything, it is an orthogonal projection.

The kernel of $P_2$ is the horizontal axis. Its image consists of points of the form $t(a,\,1)$ for any scalar $t$. This straight line is perpendicular to the horizontal axis if (and only if) $a = 0$. Thus $P_2$ is an orthogonal projection if and only if $a = 0$.

The remaining problems are from the Lecture notes on Vectors

`http://www.math.upenn.edu/~kazdan/312F12/notes/vectors/vectors8.pdf`

8. [p. 8 #5] The origin and the vectors $X$, $Y$, and $X+Y$ define a parallelogram whose diagonals have length $X+Y$ and $X-Y$. Prove the *parallelogram law*

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2;$$

This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

SOLUTION: The standard procedure is to express the norm in terms of the inner product and use the usual algebraic rules for the inner product. Thus

$$\|X+Y\|^2 = \langle X+Y,\, X+Y\rangle = \langle X,\, X\rangle+\langle X,\, Y\rangle+\langle Y,\, X\rangle+\langle Y,\, Y\rangle = \langle X,\, X\rangle+2\langle X,\, Y\rangle+\langle Y,\, Y\rangle,$$

with a similar formula for $\|X-Y\|^2$. After easy algebra, the result is clear.

9. [p. 8 #6]

   a) Find the distance $D$ from the straight line $3x-4y=10$ to the origin.

      SOLUTION: Note that the equation of the parallel line $\ell$ through the origin is $3x-4y=0$, which we rewrite as $\langle N,\, X\rangle = 0$, where $N := (3,\, -4)$ and $X = (x,\, y)$. Let $X_0$ be some point on the original line, so $\langle N,\, X_0\rangle = 10$. Then the desired distance $D$ is the same as the distance from $X_0$ to the line $\ell$: $\langle N,\, X\rangle = 0$, through the origin. But the equation for $\ell$ says the vector $N$ is perpendicular to the line $\ell$. Thus the distance $D$ is the length of the projection of $X_0$ in the direction of $N$, that is,

      $$D = \frac{|\langle N,\, X_0\rangle|}{\|N\|} = \frac{10}{5} = 2.$$

   b) Find the distance $D$ from the plane $ax+by+cz=d$ to the origin (assume the vector $\vec{N} = (a,b,c) \neq 0$).

      SOLUTION: The solution presented in the above special case generalizes immediately to give

      $$D = \frac{|\langle N,\, X_0\rangle|}{\|N\|} = \frac{|d|}{\|N\|}.$$

10. [p. 8 #8]

    a) If $X$ and $Y$ are real vectors, show that

       $$\langle X,\, Y\rangle = \frac{1}{4}\left(\|X+Y\|^2 - \|X-Y\|^2\right). \tag{3}$$

       This formula is the simplest way to recover properties of the inner product from the norm.

       SOLUTION: The straightforward procedure is the same as in Problem 8: rewrite the norms on the right in terms of the inner product and expand using algebra.

b) As an application, show that if a square matrix $R$ has the property that it preserves length, so $\|RX\| = \|X\|$ for every vector $X$, then it preserves the inner product, that is, $\langle RX,\, RY\rangle = \langle X,\, Y\rangle$ for all vectors $X$ and $Y$.

SOLUTION: We know that $\|RZ\| = \|Z\|$ for any vector $Z$. This implies $\|R(X+Y)\| = \|X+Y\|$ for any vectors $X$ and $Y$, and, similarly, $\|R(X-Y)\| = \|X-Y\|$ for any vectors $X$ and $Y$. Consequently, by equation (3) (used twice)

$$\begin{aligned} 4\langle RX,\, RY\rangle =& \|R(X+Y)\|^2 - \|R(X-Y)\|^2 \\ =& \|X+Y\|^2 - \|X-Y\|^2 \\ =& 4\langle X,\, Y\rangle \end{aligned}$$

for all vectors $X$ and $Y$.

11. [p. 9 #10]

a) If a certain matrix $C$ satisfies $\langle X,\, CY\rangle = 0$ for *all* vectors $X$ and $Y$, show that $C = 0$.

SOLUTION: Since $X$ can be *any* vector, let $X = CY$ to show that $\|CY\|^2 = \langle CY,\, CY\rangle = 0$. Thus $CY = 0$ for all $Y$ so $C = 0$.

b) If the matrices $A$ and $B$ satisfy $\langle X,\, AY\rangle = \langle X,\, BY\rangle$ for all vectors $X$ and $Y$, show that $A = B$.

SOLUTION: We have

$$0 = \langle X,\, AY\rangle - \langle X,\, BY\rangle = \langle X,\, (AY - BY)\rangle = \langle X,\, (A-B)Y\rangle$$

for all $X$ and $Y$ so by part (a) with $C := A - B$, we conclude that $A = B$.

12. [p. 9 #11–12] A matrix $A$ is called *anti-symmetric* (or skew-symmetric) if $A^* = -A$.

a) Give an example of a $3\times 3$ anti-symmetric matrix.

SOLUTION: The most general anti-symmetric $3\times 3$ matrix has the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

b) If $A$ is any anti-symmetric matrix, show that $\langle X,\, AX\rangle = 0$ for all vectors $X$.

SOLUTION: $\langle X,\, AX\rangle = \langle A^*X,\, X\rangle = -\langle AX,\, X\rangle = -\langle X,\, AX\rangle$. Thus $2\langle X,\, AX\rangle = 0$ so $\langle X,\, AX\rangle = 0$.

c) Say $X(t)$ is a solution of the differential equation $\frac{dX}{dt} = AX$, where $A$ is an anti-symmetric matrix. Show that $\|X(t)\| = \text{constant}$. [REMARK: A special case is that $X(t) := \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ satisfies $X' = AX$ with $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ so this problem gives another proof that $\cos^2 t + \sin^2 t = 1$].

SOLUTION: Let $E(t) := \|X(t)\|^2$. We show that $dE/dt = 0$. But, using part (b),

$$\frac{dE}{dt} = \frac{d}{dt}\langle X(t),\, X(t)\rangle = 2\langle X(t),\, X'(t)\rangle = 2\langle X(t),\, AX(t)\rangle = 0.$$

## Bonus Problem

[Please give this directly to Professor Kazdan]

1-B This is a followup to problem 7.

a) If a projection $P$ is self-adjoint, so $P^* = P$, show that $P$ is an orthogonal projection.

b) Conversely, if $P$ is an orthogonal projection, show that it is self-adjoint.

[Last revised: November 10, 2012]