Math 312, Fall 2012 Jerry L. Kazdan

Problem Set 6

DUE: In class Thursday, Oct. 25 *Late papers will be accepted until 1:00 PM Friday.*

REMARK: We have completed Chapter 5, Sections 5.1, 5.2, 5.3, and 5.4 (except for the QR Factorization – which we will skip). Since Fall Break interrupts this week, this assignment will be shorter.

1. [BRETSCHER, SEC. 5.2 #34] Find an orthonormal basis for the kernel of the matrix

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

SOLUTION: First we find a basis, then we find an orthonormal basis. To find the kernel of A, solve the equations

$$\begin{aligned} x_1 + x_2 \quad &+ x_3 + x_4 &= 0 \\ x_1 + 2x_2 \quad &+ 3x_3 + 4x_4 &= 0 \end{aligned}$$

say, for x_1 and x_2 in terms of x_3 and x_4. This gives $x_1 = x_3 + 2x_4$ and $x_2 = -2x_3 - 3x_4$. Thus, the vectors $\vec{x}$ in the kernel of A have the form

$$\vec{x} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} x_4 = x_3\vec{v}_1 + x_4\vec{v}_2,$$

where $\vec{v}_1$ and $\vec{v}_2$ are defined to be the vectors in the middle of the previous line. They give a basis for the kernel of A.

Next we get an orthogonal basis. The first element will be $\vec{v}_1$. For the second, we want the component of $\vec{v}_2$ that is perpendicular to $\vec{v}_1$. That is, write

$$\vec{v}_2 = a\vec{v}_1 + \vec{z} \qquad \text{where} \qquad \vec{z} \perp \vec{v}_1.$$

as usual, to find the constant a take the inner product of both sides with $\vec{v}_1$ to find $a = \langle \vec{v}_2, \vec{v}_1 \rangle / \|\vec{v}_1\|^2 = 8/6 = 4/3$. Thus,

$$\vec{z} = \vec{v}_2 - (4/3)\vec{v}_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ -4/3 \\ 1 \end{pmatrix}$$

so $\vec{v}_1$ and $\vec{z}$ are an orthogonal basis. To get an orthonormal basis, we just make these into unit vectors: $\vec{u}_1 = \vec{v}_1 / \|\vec{v}_1\|$, $\vec{u}_2 = \vec{z} / \|\vec{z}\|$.

2. [Bretscher, Sec. 5.4 #20] Using pencil and paper, find the least-squares solution to $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

Solution: We need to solve the normal equations $A^*A\vec{x} = A^*A^*\vec{b}$, that is

$$\begin{aligned} 2x_1 + x_2 &= 6 \\ x_1 + 2x_2 &= 6 \end{aligned}$$

whose solution is $x_1 = x_2 = 2$.

3. Use the Method of Least Squares to find the parabola $y = ax^2 + b$ that best fits the following data given by the following four points (x_j, y_j), $j = 1, \dots, 4$:

$$(-2, 4), \qquad (-1, 3), \qquad (0, 1), \qquad (2, 0).$$

Ideally, you'd like to pick the coefficients a and b so that the four equations $ax_j^2 + b = y_j$, $j = 1, \dots, 4$ are all satisfied. Since this probably can't be done, one uses least squares to find the best possible a and b.

Solution: Using the data, ideally we want to solve

$$\begin{aligned} (-2)^2 a + b &= 4 \\ (-1)^2 a + b &= 3 \\ 0^2 a + b &= 1 \\ 2^2 a + b &= 0 \end{aligned}, \qquad \text{that is,} \qquad \begin{pmatrix} 4 & 1 \\ 1 & 1 \\ 0 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

This has the form $A\vec{x} = \vec{b}$, where $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ etc. We need to solve the normal equations $A^*A\vec{x} = A^*\vec{b}$. Here $A^*A = \begin{pmatrix} 33 & 9 \\ 9 & 4 \end{pmatrix}$, etc. This is routine.

4. The water level in the North Sea is mainly determined by the so-called M2 tide, whose period is about 12 hours. The height $H(t)$ thus roughly has the form

$$H(t) = c + a\sin(2\pi t/12) + b\cos(2\pi t/12),$$

where time t is measured in hours (note $\sin(2\pi t/12$ and $\cos(2\pi t/12)$ are periodic with period 12 hours). Say one has the following measurements:

t (hours)	0	2	4	6	8	10
$H(t)$ (meters)	1.0	1.6	1.4	0.6	0.2	0.8

Use the method of least squares to find the constants a, b, and c in $H(t)$ for this data.

SOLUTION: Using the data, the equations we ideally wish to solve are

$$\begin{aligned}
c + a\sin 0 \quad &+ c\cos 0 &&= 1.0 \\
c + a\sin \pi/3 \quad &+ c\cos \pi/3 &&= 1.6 \\
c + a\sin 2\pi/3 \quad &+ c\cos 2\pi/3 &&= 1.4 \\
c + a\sin \pi \quad &+ c\cos \pi &&= 0.6 \\
c + a\sin 4\pi/3 \quad &+ c\cos 4\pi/3 &&= 0.2 \\
c + a\sin 5\pi/3 \quad &+ c\cos 5\pi/3 &&= 0.8
\end{aligned}, \quad \text{that is,} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & \sqrt{3}/2 & 1/2 \\ 1 & \sqrt{3}/2 & -1/2 \\ 1 & 0 & -1 \\ 1 & -\sqrt{3}/2 & -1/2 \\ 1 & -\sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} c \\ a \\ b \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.6 \\ 1.4 \\ 0.6 \\ 0.2 \\ 0.8 \end{pmatrix}$$

This is now in standard form for the method of least squares. I used Maple and found: $c = 0.93$, $a = 0.58$, and $b = 0.27$, so

$$H(t) = 0.93 + 0.58\sin(2\pi t/12) + 0.27\cos(2\pi t/12),$$

5. Let A be a real matrix, not necessarily square.

a) Show that both A^*A and AA^* are self-adjoint.

SOLUTION: Use $(AB)^* = B^*A^*$ and $(A^*)^* = A$. This is easy. The example where $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is illuminating: $A^*A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\quad AA^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b) Show that $\ker A = \ker A^*A$. [HINT: Show separately that $\ker A \subset \ker A^*A$ and $\ker A^*A \subset \ker A$. The identity $\langle \vec{x},\, A^*A\vec{x}\rangle = \langle A\vec{x},\, A\vec{x}\rangle$ is useful.]

SOLUTION: If $\vec{x} \in \ker A$, then $A\vec{x} = 0$ so $A^*A\vec{x} = A^*0 = 0$. Thus $\vec{x} \in \ker A^*A$. In other words, $\ker A \subset \ker A^*A$.

Conversely, if $\vec{x} \in \ker A^*A$, then $A^*A\vec{x} = 0$ so

$$0 = \langle \vec{x},\, A^*A\vec{x}\rangle = \langle A\vec{x},\, A\vec{x}\rangle = \|A\vec{x}\|^2.$$

Consequently $A\vec{x} = 0$, that is, $\vec{x} \in \ker A$. This proves that $\ker A^*A \subset \ker A$.

[Last revised: October 25, 2012]