Problem Set 8

Due: In class Thursday, Nov. 8 *Late papers will be accepted until 1:00 PM Friday.*

Some of this is on the material in Bretscher, Sec. 5.5, concerning inner products in spaces of functions. *No* new ideas are involved, but it does take time to simply relax.

1. For a square matrix A, a scalar λ is an *eigenvalue* and a vector $\vec{v} \neq 0$ is a corresponding *eigenvector* if $A\vec{v} = \lambda\vec{v}$, so A maps $\vec{v}$ to a multiple of itself.

   If A is a symmetric (that is, self-adjoint) matrix with eigenvalues λ, μ, $\lambda \neq \mu$ and corresponding eigenvectors $\vec{v}$ and $\vec{w}$. Show that $\vec{v}$ and $\vec{w}$ are orthogonal.

2. Introduce the following inner product on the space of continuous functions on the interval $-1 \leq x \leq 1$: $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.
   a) Show that $1 \perp x$.
   b) For which constants a, b is $f(x) := a + bx + x^2$ orthogonal to both 1 and x?
   c) Find an orthogonal basis for the span of 1, x, and x^2.

3. A real-valued function is called *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$ for all x. For instance, $2x^4 + x\sin 3x$ is even and $\sin 4x - 7x^5$ is odd. Using the same inner product as above,
   a) Show that any odd function $f(x)$ is orthogonal to the function 1.
   b) Show that any even function $f(x)$ is orthogonal to $\sin 13x$.
   c) Show that the product of an even function $f(x)$ and an odd function $g(x)$ is odd.
   d) Show that any even function $f(x)$ is orthogonal to any odd function $g(x)$.

4. [Bretscher, Sec. 5.5 #16] Consider the space of continuous functions on the interval [0, 1] (that is, $0 \leq x \leq 1$) with the inner product $\langle f, g \rangle := \int_{0}^{1} f(x)g(x)\,dx$.
   a) Using this inner product, find an orthonormal basis for the space $\mathcal{P}_1$ of polynomials of degree at most one.
   b) Find a linear polynomial $g(x) = a + bx$ that best approximates x^2 in the norm defined by this inner product.

5. [Bretscher, Sec. 5.5 #20]. In $\mathbb{R}^2$ consider the NEW inner product $\ll \vec{v}, \vec{w} \gg := \vec{v}^T \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \vec{w}$ with corresponding norm $|||\vec{v}|||^2 := \ll \vec{v}, \vec{v} \gg$.
   a) Find all vectors in $\mathbb{R}^2$ that are orthogonal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
   b) Find an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product.

6. [Bretscher, Sec. 5.5 #24]. Using the inner product of problem 4, for the polynomials $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ say we are given the following table of inner products:

$\langle\ ,\ \rangle$	$\mathbf{f}$	$\mathbf{g}$	$\mathbf{h}$
$\mathbf{f}$	4	0	8
$\mathbf{g}$	0	1	3
$\mathbf{h}$	8	3	50

For example, $\langle \mathbf{g}, \mathbf{h}\rangle = \langle \mathbf{h}, \mathbf{g}\rangle = 3$. Let E be the span of $\mathbf{f}$ and $\mathbf{g}$.

a) Compute $\langle \mathbf{f}, \mathbf{g}+\mathbf{h}\rangle$.

b) Compute $\|\mathbf{g}+\mathbf{h}\|$.

c) Find $\text{proj}_E \mathbf{h}$. [Express your solution as linear combinations of $\mathbf{f}$ and $\mathbf{g}$.]

d) Find an orthonormal basis of the span of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ [Express your results as linear combinations of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$.]

7. [Like Bretscher, Sec. 5.5 #26 & 28]. Use the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Define

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x \le 0, \\ 1 & \text{if } 0 < x \le \pi, \end{cases}$$

and extend f to all of $\mathbb{R}$ as period is with period 2π: $f(x+2\pi) = f(x)$. This is called a *square wave.*

a) Compute the first N terms in the Fourier Series

$$f(x) = A_0 + \sum_{k=1}^{N} [A_k \cos kx + B_k \sin kx]$$

b) Apply the Pythagorean Theorem 5.5.6 (page 343) to your answer.

8. Compute the determinant of the upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

[Do the cases $n = 2$ and $n = 3$ first.]

9. The $n \times n$ matrices A and B are *similar* if there is and invertible $n \times n$ matrix S so that $B = SAS^{-1}$. If A and B are similar, show that $\det B = \det A$.

[Last revised: November 14, 2012]