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Signature

PRINTED NAME

Math 312  
Feb. 18, 2014

## Exam 1

Jerry L. Kazdan  
9:00 – 10:20

DIRECTIONS This exam has two parts. Part A has 5 shorter questions, (10 points each so total 50 points) while Part B had 5 problems (18 points each, so total is 90 points). Maximum score is thus 140 points.

Closed book, no calculators or computers– but you may use one  $3'' \times 5''$  card with notes on both sides. *Clarity and neatness count.*

PART A: Five short answer questions (10 points each, so 50 points).

A-1. Which of the following sets are linear spaces? [If not, why not?]

a) The set of points  $(x, y) \in \mathbb{R}^2$  with  $y = 2x + x^2$ .

b) The set of once differentiable solutions  $u(x)$  of  $u' + 3x^2u = 0$ . [You are *not* being asked to solve this equation.]

c) The set of all polynomials  $p(x)$  with the property that  $\int_0^1 p(x) e^x dx = 0$ .

d) The set of  $3 \times 2$  matrices  $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$  with  $a + 2e = 0$ .

<i>Score</i>	
A-1	
A-2	
A-3	
A-4	
A-5	
B-1	
B-2	
B-3	
B-4	
B-5	
<i>Total</i>	

A-2. Let  $S$  and  $T$  be linear spaces and  $L : S \rightarrow T$  be a linear map. Say  $\vec{v}_1$  and  $\vec{v}_2$  are (distinct!) solutions of the equations  $L\vec{x} = \vec{y}_1$  while  $\vec{w}$  is a solution of  $L\vec{x} = \vec{y}_2$ . Answer the following in terms of  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{w}$ .

a) Find some solution of  $L\vec{x} = 2\vec{y}_1 - 2\vec{y}_2$ .

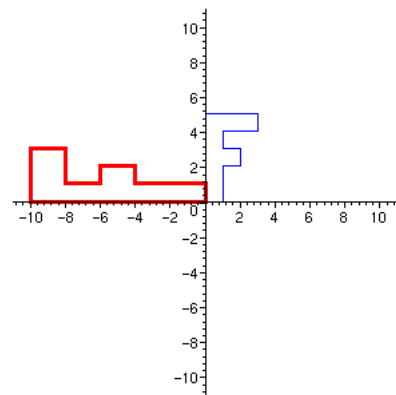
b) Find another solution (other than  $\vec{w}$ ) of  $L\vec{x} = \vec{y}_2$ .

A-3. Let  $A$  be *any*  $5 \times 3$  matrix so  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^5$  is a linear transformation. Answer the following include a brief explanation.

a) Is  $A\vec{x} = \vec{b}$  necessarily solvable for any  $\vec{b}$  in  $\mathbb{R}^5$ ?

b) Suppose the kernel of  $A$  is one dimensional. What is the dimension of the image of  $A$ ?

A-4. Find a  $2 \times 2$  matrix  $A$  that in the standard basis does the indicated transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



A-5. In  $\mathbb{R}^n$  (or any linear space with an inner product), If  $X$  and  $Y$  are orthogonal, show that the Pythagorean Theorem holds:

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2.$$

PART B Five questions, 18 points each (so 90 points total).

B-1. In  $\mathbb{R}^3$ , find the distance between the point  $P = (1, 3, -1)$  and the plane of points  $(x, x_2, x_3)$  whose coordinates satisfy  $2x_1 + x_2 - 2x_3 = 0$ .

B-2. Let  $A$  and  $B$  be  $n \times n$  real matrices. If the matrix  $BA$  is invertible, prove that both  $A$  and  $B$  are invertible.

B-3. Let the linear map  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be specified by the matrix  $A := \begin{pmatrix} 3 & 0 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$ .

a) Find a basis for the kernel of  $A$ .

b) Find a basis for the image of  $A$ .

c) With the above matrix  $A$ , is it possible to find an invertible  $3 \times 3$  matrix  $B$  so that the matrix  $AB$  is invertible? (Why?)

B-4. Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation that sends

$$\vec{e}_1 \text{ to } \vec{e}_1 + \vec{e}_3, \quad \vec{e}_2 \text{ to } -\vec{e}_1, \quad \text{and} \quad \vec{e}_3 \text{ to } \vec{e}_2 + \vec{e}_3$$

(here the  $e_j$  are the standard basis vectors).

a) Find the matrix representation of  $T$  (using the standard basis).

b) Describe what the inverse transformation  $T^{-1}$  does to each of the vectors  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ . (This will involve some computations).

c) Find all solutions  $\vec{x}$  of the equation  $T(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ .

B-5. Let  $\mathcal{P}_N$  be the linear space of polynomials of degree at most  $N$  and  $L : \mathcal{P}_N \rightarrow \mathcal{P}_N$  the linear map defined by  $Lu := u'' + bu' + cu$ , where  $b$ , and  $c$  are constants. Assume  $c \neq 0$ .

a) Compute  $L(x^k)$ .

b) In the special case  $N = 2$ , show that the kernel of  $L : \mathcal{P}_2 \rightarrow \mathcal{P}_2$  is 0. [This uses  $c \neq 0$ .]

c) Show that for every polynomial  $q(x) \in \mathcal{P}_2$  there is one (and only one) solution  $p(x) \in \mathcal{P}_2$  of the differential equation  $Lp = q$ .

d) In the general case where  $N \geq 0$  can be any integer, show that the kernel of  $L : \mathcal{P}_N \rightarrow \mathcal{P}_N$  is 0.

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- e) Show that for every polynomial  $q(x) \in \mathcal{P}_N$  there is one (and only one) solution  $p(x) \in \mathcal{P}_N$  of  $Lp = q$ . In other words, if  $c \neq 0$ , the map  $L\mathcal{P}_N \rightarrow \mathcal{P}_N$  is invertible.