Math 312 **Exam 1** Jerry L. Kazdan
Feb. 18, 2014 9:00 – 10:20

Directions This exam has two parts. Part A has 5 shorter questions, (10 points each so total 50 points) while Part B had 5 problems (18 points each, so total is 90 points). Maximum score is thus 140 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

Part A: Five short answer questions (10 points each, so 50 points).

A–1. Which of the following sets are linear spaces? [If not, why not?]

a) The set of points $(x,\, y) \in \mathbb{R}^2$ with $y = 2x + x^2$.

Solution No. $(1,3)$ is in this set but $(2,6)$ not.

b) The set of once differentiable solutions $u(x)$ of $u' + 3x^2 u = 0$. [You are *not* being asked to solve this equation.]

Solution Yes.

c) The set of all polynomials $p(x)$ with the property that $\int_0^1 p(x)\, e^x\, dx = 0$.

Solution Yes.

d) The set of 3×2 matrices $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ with $a + 2e = 0$.

Solution Yes.

A–2. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say $\vec{v}_1$ and $\vec{v}_2$ are (distinct!) solutions of the equations $L\vec{x} = \vec{y}_1$ while $\vec{w}$ is a solution of $L\vec{x} = \vec{y}_2$. Answer the following in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$.

a) Find some solution of $L\vec{x} = 2\vec{y}_1 - 2\vec{y}_2$.

Solution $\vec{x} = 2\vec{v}_1 - 2\vec{w}$

b) Find another solution (other than $\vec{w}$) of $L\vec{x} = \vec{y}_2$.

Solution $\vec{x} = \vec{v}_1 - \vec{v}_2 + \vec{w}$

A–3. Let A be *any* 5×3 matrix so $A : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation. Answer the following include a brief explanation.

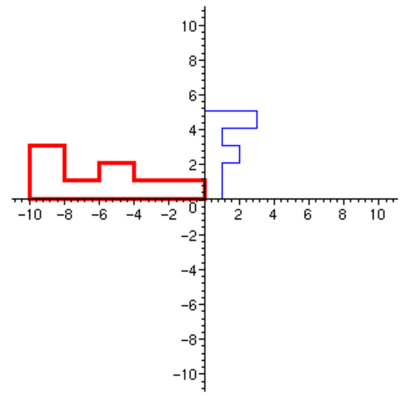
a) Is $A\vec{x} = \vec{b}$ necessarily solvable for any $\vec{b}$ in $\mathbb{R}^5$?

Solution No. For instance, if A is the zero matrix the only point in the image is $b = 0$. In addition, for this case since the dimension of the image is at most 3, there is *no* such matrix that is onto.

b) Suppose the kernel of A is one dimensional. What is the dimension of the image of A?

Solution $\dim(\ker A) = 1$ so $\dim(\mathrm{im} A) = 2$.

A–4. Find a 2×2 matrix A that in the standard basis does the indicated transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



Solution $A = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$

A–5. In $\mathbb{R}^n$ (or any linear space with an inner product), If X and Y are orthogonal, show that the Pythagorean Theorem holds:

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2.$$

Solution $\langle X + Y, X + Y, =\rangle \langle X,\, X\rangle + \langle Y,\, Y\rangle + \langle X,\, Y\rangle + \langle Y,\, X\rangle = \|X\|^2 + \|Y\|^2$

Part B Five questions, 18 points each (so 90 points total).

B–1. In $\mathbb{R}^3$, find the distance between the point $P = (1,\, 3,\, -1)$ and the plane of points $(x_{,}\, x_2,\, x_3)$ whose coordinates satisfy $2x_1 + x_2 - 2x_3 = 0$.

Solution The unit normal vector to this plane is $\vec{N} = \frac{(2,1,-2)}{\|(2,1,-2)\|} = (2/3,\, 1/3,\, -2/3)$ and the distance that we want is $|\langle P,\, \vec{N}\rangle| = |2/3 + 1 + 2/3| = 7/3$

B–2. Let A and B be $n \times n$ real matrices. If the matrix $C := BA$ is invertible, prove that both A and B are invertible.

Solution Since BA invertible we have $ker(BA) = 0$. Also $\ker(A) \subset ker(BA)$ so $\ker(A) = 0$. Because A is a square matrix it is invertible.

Let $C = BA$. Then $B = CA^{-1}$ so B is the product of two invertible matrices and hence is invertible. [Alternate: Since $\operatorname{im}(BA) \subset \operatorname{im}(B)$ and BA is onto, then B is onto. Because $B$is square it is invertible.]

B–3. Let the linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ be specified by the matrix $A := \begin{pmatrix} 3 & 0 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$.

a) Find a basis for the kernel of A.

SOLUTION After solving $A\vec{x} = \vec{0}$ we obtain $x_3 = -3x_1$, $x_2 = -x_1$ hence $ker(A) = \{(t, -t, -3t) | t \in \mathbb{R}\} = \text{span}\{(1, -1, -3)\}$.

b) Find a basis for the image of A.

SOLUTION From (a) and the rank-nullity theorem we have that $\dim(\text{im}\, A) = 2$. Also we know that the image of A is the space of columns of A. Examining the columns of A we see that the first and second column of A are linearly independent hence they form a basis for $\text{im}(A)$ i.e. $\{(3, 1, 2), (0, -2, -1)\}$ is a basis for $im(A)$. There are many other obvious bases, such as the first and third columns of A.

c) With the above matrix A, is it possible to find an invertible 3×3 matrix B so that the matrix AB is invertible? (Why?)

SOLUTION No. Since $\text{im}(AB) \subset \text{im}A$ and the image of A has dimension 2, then AB cannot be onto and hence cannot be invertible.

[ALTERNATE: Suppose yes. Then there exists an invertible 3×3 matrix B such that AB invertible. Since A, B are 3×3 matrices and AB is invertible, then problem B-2 implies that A is invertible. So we have a contradiction namely there does not exist such matrix B.

B–4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that sends

$$\vec{e}_1 \quad \text{to} \quad \vec{e}_1 + \vec{e}_3, \qquad \vec{e}_2 \quad \text{to} \quad -\vec{e}_1, \qquad \text{and} \qquad \vec{e}_3 \quad \text{to} \quad \vec{e}_2 + \vec{e}_3$$

(here the e_j are the standard basis vectors).

a) Find the matrix representation of T (using the standard basis).

SOLUTION From the information give we have that the matrix has as first column $\vec{e}_1 + \vec{e}_3$, as second column $-\vec{e}_2$ and as third column $\vec{e}_2 + \vec{e}_3$. Hence:

$$T = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

b) Describe what the inverse transformation T^{-1} does to each of the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. (This will involve some computations).

SOLUTION Since $T(\vec{e}_1) = \vec{e}_1 + \vec{e}_3$, $T(\vec{e}_2) = -\vec{e}_1$, and $T(\vec{e}_3) = \vec{e}_2 + \vec{e}_3$, we have that:

$$T^{-1}(\vec{e}_1 + \vec{e}_3) = \vec{e}_1, \qquad T^{-1}(-\vec{e}_1) = \vec{e}_2, \qquad \text{and} \qquad T^{-1}(\vec{e}_2 + \vec{e}_3) = \vec{e}_3$$

hence,

$$T^{-1}(\vec{e}_1) = -\vec{e}_2$$
$$T^{-1}(\vec{e}_3) = T^{-1}((\vec{e}_1 + \vec{e}_3) - \vec{e}_1) = T^{-1}(\vec{e}_1 + \vec{e}_3) - T^{-1}(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$$
$$T^{-1}(\vec{e}_2) = T^{-1}((\vec{e}_2 + \vec{e}_3) - \vec{e}_3) = T^{-1}(\vec{e}_2 + \vec{e}_3) - T^{-1}(\vec{e}_3) = \vec{e}_3 - \vec{e}_1 - \vec{e}_2$$

ALTERNATE: Use any method to compute the inverse of the matrix you found in part a).

c) Find all solutions $\vec{x}$ of the equation $T(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

SOLUTION Since $\vec{x} = T^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = T^{-1}(\vec{e}_1) + 2T^{-1}(\vec{e}_2) + 3T^{-1}(\vec{e}_3)$, we have that:

$$\vec{x} = -\vec{e}_2 + 2\vec{e}_3 - 2\vec{e}_1 - 2\vec{e}_2 + 3\vec{e}_1 + 3\vec{e}_2 = \vec{e}_1 + 2\vec{e}_3$$

B–5. Let $\mathcal{P}_N$ be the linear space of polynomials of degree at most N and $L : \mathcal{P}_N \to \mathcal{P}_N$ the linear map defined by $Lu := u'' + bu' + cu$, where b, and c are constants. Assume $c \neq 0$.

a) Compute $L(x^k)$.

SOLUTION $L(x^k) = cx^k + bkx^{k-1} + k(k-1)x^{k-2}$.

b) In the special case $N = 2$, show that the kernel of $L : \mathcal{P}_2 \to \mathcal{P}_2$ is 0. [This uses $c \neq 0$.]

SOLUTION Say $p(x) := a_2x^2 + a_1x + a_0 \in \ker(L)$. Then

$$0 = Lp = 2a_2 + b(2a_2x + a_1) + c(a_2x^2 + a_1x + a_0) = 0.$$

That is,

$$ca_2x^2 + (2ba_2 + ca_1)x + (2a_2 + ba_1 + ca_0) = 0.$$

The coefficients of each power of x must be zero. In particular, $ca_2 = 0$. Since $c \neq 0$, then $a_2 = 0$. Looking at the coefficient of x this then implies that $a_1 = 0$. Finally, $2a_2 + ba_1 + ca_0 = 0$ then implies $a_0 = 0$.

c) Show that for every polynomial $q(x) \in \mathcal{P}_2$ there is one (and only one) solution $p(x) \in \mathcal{P}_2$ of the differential equation $Lp = q$.

SOLUTION Since L has trivial kernel then by the rank-nullity theorem $L : \mathcal{P}_2 \to \mathcal{P}_2$ is also onto and hence invertible. Thus for all $q \in \mathcal{P}_2$ there exists a unique $p \in \mathcal{P}_2$ such that $Lp = q$.

d) In the general case where $N \geq 0$ can be any integer, show that the kernel of $L : \mathcal{P}_N \to \mathcal{P}_N$ is 0.

SOLUTION Say $p(x) = a_kx^k + a_{k-1}x^{k-1} + \cdots + a_1x + a_0$ is in the kernel of L. Assume that $p(x)$ is not the zero polynomial and actually has degree k so $a_k \neq 0$. Computing Lp we obtain a polynomial of degree k. By part a)

$$0 = Lp = (ca_k)x^k + \text{lower order terms}.$$

Thus $ca_k = 0$. Because $c \neq 0$ then $a_k = 0$. This contradicts $a_k \neq 0$. Thus the only possibility for $Lp = 0$ is that p is the zero polynomial.

e) Show that for every polynomial $q(x) \in \mathcal{P}_N$ there is one (and only one) solution $p(x) \in \mathcal{P}_N$ of $Lp = q$. In other words, if $c \neq 0$, the map $L\mathcal{P}_N \to \mathcal{P}_N$ is invertible.

SOLUTION Uniqueness: If $p_1, p_2 \in \mathcal{P}_N$ solutions then $L(p_1 - p_2) = Lp_1 - Lp_2 = q - q = 0$ so $p_1 - p_2 \in \ker(L)$ hence $p_1 = p_2$.

Existence: Since N is fixed, $\dim \mathcal{P}_N$ is finite. Because $\ker L = 0$, by the rank-nullity theorem L is onto and hence invertible.