Math 312 — **Exam 1** — Jerry L. Kazdan

Feb. 18, 2014 — 9:00 – 10:20

Directions This exam has two parts. Part A has 5 shorter questions, (10 points each so total 50 points) while Part B had 5 problems (18 points each, so total is 90 points). Maximum score is thus 140 points.

Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

Part A: Five short answer questions (10 points each, so 50 points).

A–1. Which of the following sets are linear spaces? [If not, why not?]

a) The set of points $(x,\, y) \in \mathbb{R}^2$ with $y = 2x + x^2$.

b) The set of once differentiable solutions $u(x)$ of $u' + 3x^2 u = 0$. [You are *not* being asked to solve this equation.]

c) The set of all polynomials $p(x)$ with the property that $\int_0^1 p(x)\, e^x\, dx = 0$.

d) The set of 3×2 matrices $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ with $a + 2e = 0$.

A–2. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say $\vec{v}_1$ and $\vec{v}_2$ are (distinct!) solutions of the equations $L\vec{x} = \vec{y}_1$ while $\vec{w}$ is a solution of $L\vec{x} = \vec{y}_2$. Answer the following in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$.

a) Find some solution of $L\vec{x} = 2\vec{y}_1 - 2\vec{y}_2$.

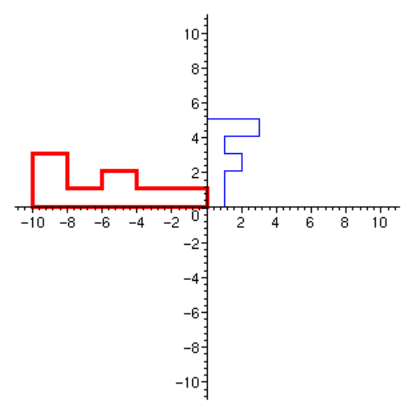
b) Find another solution (other than $\vec{w}$) of $L\vec{x} = \vec{y}_2$.

A–3. Let A be *any* 5×3 matrix so $A : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation. Answer the following include a brief explanation.

a) Is $A\vec{x} = \vec{b}$ necessarily solvable for any $\vec{b}$ in $\mathbb{R}^5$?

b) Suppose the kernel of A is one dimensional. What is the dimension of the image of A?

A–4. Find a 2×2 matrix A that in the standard basis does the indicated transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



A–5. In $\mathbb{R}^n$ (or any linear space with an inner product), If X and Y are orthogonal, show that the Pythagorean Theorem holds:

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2.$$

Part B Five questions, 18 points each (so 90 points total).

B–1. In $\mathbb{R}^3$, find the distance between the point $P = (1,\ 3,\ -1)$ and the plane of points $(x_,\, x_2,\, x_3)$ whose coordinates satisfy $2x_1 + x_2 - 2x_3 = 0$.

B–2. Let A and B be $n \times n$ real matrices. If the matrix $C := BA$ is invertible, prove that both A and B are invertible.

B–3. Let the linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ be specified by the matrix $A := \begin{pmatrix} 3 & 0 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$.

a) Find a basis for the kernel of A.

b) Find a basis for the image of A.

c) With the above matrix A, is it possible to find an invertible 3×3 matrix B so that the matrix AB is invertible? (Why?)

B–4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that sends

$$\vec{e}_1 \quad \text{to} \quad \vec{e}_1 + \vec{e}_3, \qquad \vec{e}_2 \quad \text{to} \quad -\vec{e}_1, \qquad \text{and} \qquad \vec{e}_3 \quad \text{to} \quad \vec{e}_2 + \vec{e}_3$$

(here the e_j are the standard basis vectors).

a) Find the matrix representation of T (using the standard basis).

b) Describe what the inverse transformation T^{-1} does to each of the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. (This will involve some computations).

c) Find all solutions $\vec{x}$ of the equation $T(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

B–5. Let $\mathcal{P}_N$ be the linear space of polynomials of degree at most N and $L : \mathcal{P}_N \to \mathcal{P}_N$ the linear map defined by $Lu := u'' + bu' + cu$, where b, and c are constants. Assume $c \neq 0$.

a) Compute $L(x^k)$.

b) In the special case $N = 2$, show that the kernel of $L : \mathcal{P}_2 \to \mathcal{P}_2$ is 0. [This uses $c \neq 0$.]

c) Show that for every polynomial $q(x) \in \mathcal{P}_2$ there is one (and only one) solution $p(x) \in \mathcal{P}_2$ of the differential equation $Lp = q$.

d) In the general case where $N \geq 0$ can be any integer, show that the kernel of $L : \mathcal{P}_N \to \mathcal{P}_N$ is 0.

e) Show that for every polynomial $q(x) \in \mathcal{P}_N$ there is one (and only one) solution $p(x) \in \mathcal{P}_N$ of $Lp = q$. In other words, if $c \neq 0$, the map $L\mathcal{P}_N \to \mathcal{P}_N$ is invertible.