ODE-Diagonalize: Examples

EXAMPLE 1 Let $A := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Solve

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \text{with} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1}$$

SOLUTION: The key observation is that if A were a diagonal matrix, this would be simple. Thus we begin by finding the eigenvalues ad eigenvectors of A. By an easy calculation

$$\det(A - \lambda I) = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5).$$

Thus the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. From here we can proceed in two slightly different ways.

METHOD 1 Observe that A is similar to the diagonal matrix $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, that is, $S^{-1}AS = D$, where $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ has the corresponding eigenvectors as its columns. Thus $A = SDS^{-1}$.
We now use this in our differential equation: $\vec{x}'(t) = SDS^{-1}\vec{x}$. Multiply both sides by S^{-1}. Since S does not depend on t, $(S^{-1}\vec{x}(t))' = DS^{-1}\vec{x}$. This is simpler to use if we let $\vec{y}(t) = S^{-1}\vec{x}$. Then the differential equation becomes

$$\frac{d\vec{y}(t)}{dt} = D\vec{y}(t),$$

that is,

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 3y_1(t) \\ 5y_2(t) \end{pmatrix}.$$

These are *uncoupled* differential equations, $y_1' = 3y_1$, $y_2' = 4y_2$, that one can solve immediately giving

$$y_1(t) = ae^{3t}, \qquad y_2(t) = be^{5t},$$

for any constants a and b.
It remains to return to restate this in terms of $\vec{x}(t)$

$$\vec{x}(t) = S\vec{y}(t) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} ae^{3t} \\ be^{5t} \end{pmatrix} = \begin{pmatrix} ae^{3t} + be^{5t} \\ -ae^{3t} + be^{5t} \end{pmatrix}$$

We use the initial condition to determine the constants a and b.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} a + b \\ -a + b \end{pmatrix}.$$

Thus $a = b = 1/2$. Therefore

$$\vec{x}(t) = \frac{1}{2} \begin{pmatrix} e^{3t} + e^{5t} \\ -e^{3t} + e^{5t} \end{pmatrix}.$$

METHOD 2 Since the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are a basis for $\mathbb{R}^2$, given any $\vec{x}(t)$, there are functions $y_1(t)$ and $y_2(t)$ so that

$$\vec{x}(t) = y_1(t)\vec{v}_1 + y_2(t)\vec{v}_2. \tag{2}$$

We now plug this in the differential equation $\vec{x}\,' = A\vec{x}$. The left side becomes

$$\vec{x}\,'(t) = y_1'(t)\vec{v}_1 + y_2'(t)\vec{v}_2,$$

and the more interesting right side becomes

$$A\vec{x} = 3y_1\vec{v}_1 + 5y_2\vec{v}_2.$$

Comparing the coefficients of $\vec{v}_1$ and $\vec{v}_2$ in the last two equations we conclude that

$$y_1' = 3y_1 \qquad \text{and} \qquad y_2' = 5y_2.$$

Their solutions are

$$y_1(t) = ae^{3t} \qquad \text{and} \qquad y_2(t) = be^{5t}$$

for any constants a and b. Using this in equation (2) we find

$$\vec{x}(t) = ae^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + be^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Finally, use the initial condition to determine a and b:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(0) = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives $a = b = 1/2$. Therefore

$$\vec{x}(t) = \frac{1}{2}\left[e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right].$$

EXAMPLE 2 Let $A := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Solve

$$\frac{d^2\vec{x}}{dt^2} = A\vec{x} \qquad \text{with} \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{x}\,'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

SOLUTION: This is the sameas equation (1) except that here we have a second derivative. Both of the methods used in Example 1 work here with essentially no change. We'll use Method 2. Since the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A are a basis for $\mathbb{R}^2$, given any $\vec{x}(t)$, there are functions $y_1(t)$ and $y_2(t)$ so that

$$\vec{x}(t) = y_1(t)\vec{v}_1 + y_2(t)\vec{v}_2. \tag{4}$$

We now plug this in the differential equation $\vec{x}\,'' = A\vec{x}$. The left side becomes

$$\vec{x}\,''(t) = y_1''(t)\vec{v}_1 + y_2''(t)\vec{v}_2,$$

and the more interesting right side becomes

$$A\vec{x} = 3y_1\vec{v}_1 + 5y_2\vec{v}_2.$$

Comparing the coefficients of $\vec{v}_1$ and $\vec{v}_2$ in the last two equations we conclude that

$$y_1'' = 3y_1 \qquad \text{and} \qquad y_2'' = 5y_2.$$

Both of these equations have the form $u'' = k^2u$ whose general solution is

$$u(t) = c_1e^{kt} + c_2e^{-kt}.$$

Thus

$$\begin{aligned} y_1(t) =& ae^{\sqrt{3}\,t} + be^{-\sqrt{3}\,t} \\ y_2(t) =& ce^{\sqrt{5}\,t} + de^{-\sqrt{5}\,t} \end{aligned}$$

for any choice of the constants a, b, c and d. Plug this into equation (4) to find

$$\vec{x}(t) = (ae^{\sqrt{3}\,t} + be^{-\sqrt{3}\,t})\begin{pmatrix}1\\-1\end{pmatrix} + (ce^{\sqrt{5}\,t} + de^{-\sqrt{5}\,t})\begin{pmatrix}1\\1\end{pmatrix}.$$

We use the initial conditions to determine the constants a, b, c and d:

$$\begin{pmatrix}2\\0\end{pmatrix} = \vec{x}(0) = (a+b))\begin{pmatrix}1\\-1\end{pmatrix} + (c+d))\begin{pmatrix}1\\1\end{pmatrix},$$

and

$$\begin{pmatrix}0\\0\end{pmatrix} = \vec{x}\,'(0) = (a-b)\sqrt{3}\begin{pmatrix}1\\-1\end{pmatrix} + (c-d)\sqrt{5}\begin{pmatrix}1\\1\end{pmatrix}.$$

Therefore, by a routine computation, $a = b = c = d = \frac{1}{2}$ so

$$\begin{aligned} \vec{x}(t) =& \frac{1}{2}(e^{\sqrt{3}\,t} + e^{-\sqrt{3}\,t})\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{2}(e^{\sqrt{5}\,t} + e^{-\sqrt{5}\,t})\begin{pmatrix}1\\1\end{pmatrix} \\ =& \cosh(\sqrt{3}\,t)\begin{pmatrix}1\\-1\end{pmatrix} + \cosh(\sqrt{5}\,t)\begin{pmatrix}1\\1\end{pmatrix} \\ =& \begin{pmatrix}\cosh(\sqrt{3}\,t) + \cosh(\sqrt{5}\,t)\\ -\cosh(\sqrt{3}\,t) + \cosh(\sqrt{5}\,t)\end{pmatrix} \end{aligned}$$

Note that *any* of the last three lines are valid formulas for the solution $\vec{x}(t)$, Your preference depends on what you will next do with the solution.