

$A : \mathbb{R}^n \rightarrow \mathbb{R}^k$, a Linear Map

Fact 1 The following are equivalent:

- A is one-to-one.
- $\ker(A) = 0$.
- $\dim \ker(A) = 0$.
- The equation $Ax = y$ has at most one solution.
- The columns of A are linearly independent.
- The rows of A span \mathbb{R}^n .
- A^T is onto.

Fact 2 The following are equivalent:

- A is onto.
- $\text{image}(A) = \mathbb{R}^k$.
- $\dim \text{image}(A) = k$.
- $\text{rank}(A) = k$.
- The equation $Ax = y$ has at least one solution.
- The rows of A are linearly independent.
- The columns of A span \mathbb{R}^k .
- A^T is one-to-one.

Fact 3 If $n = k$ the following are equivalent:

- A is invertible.
- Everything in Fact 1.
- Everything in Fact 2.
- For every y there is exactly one solution of $Ax = y$
- A is bijective (equivalently, A is an isomorphism).
- 0 is not an eigenvalue of A .
- $\det(A) \neq 0$