Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 1

DUE: In class Thursday, Jan. 23. *Late papers will be accepted until 1:00 PM Friday.*

These problems are intended to be straightforward with not much computation. I warmly suggest that you discuss these problems with others in the class.

REMARK: Before class on Tuesday please read *all* of Chapters 1 and 2 in the text. Essentially it should be a review from Math 240.

1. Let $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$. Compute the inverse of A and of A^2.

2. Solve all of the following equations. [Note that the left sides of these equations are identical.]

   a). $\begin{aligned} 2x + 5y &= \ \ 5 \\ x + 3y &= -1 \end{aligned}$ b). $\begin{aligned} 2x + 5y &= \ \ 0 \\ x + 3y &= -2 \end{aligned}$ c). $\begin{aligned} 2x + 5y &= 1 \\ x + 3y &= 0 \end{aligned}$ d). $\begin{aligned} 2x + 5y &= 2 \\ x + 3y &= 1 \end{aligned}$

3. [Bretscher, Sec.2.1 #13] Finding the inverse of a matrix A means solving the system of equations $A\vec{x} = \vec{y}$ for $\vec{x}$, so $\vec{x} = A^{-1}\vec{y}$.
   a) Let $A := \left(\begin{smallmatrix} 1 & 2 \\ c & 6 \end{smallmatrix}\right)$. With your bare hands (as on page 2 of the textbook – not using anything about determinants) show that A is invertible if and only if $c \neq 3$.
   b) Let $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. With your bare hands (not using anything about determinants) show that M is invertible if and only if $ad - bc \neq 0$. [*Hint:* Treat the cases $a \neq 0$ and $a = 0$ separately.]

4. Let A and B be 2×2 matrices.
   a) If B is invertible and $AB = 0$, show that $A = 0$.
   b) Give an example where $AB = 0$ but $BA \neq 0$.
   c) Find an example of a 2×2 matrix with the property that $A^2 = 0$ but $A \neq 0$.
   d) Find all *invertible* $n \times n$ matrices A with the property $A^2 = 3A$.

5. [Bretscher, Sec.2.3 #19] Find all the matrices that commute with $A := \left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$.

6. a) Find a real 2×2 matrix A (other than $A = \pm I$) such that $A^2 = I$.
   b) Find a real 2×2 matrix A such that $A^4 = I$ but $A^2 \neq I$.

7. Let L, M, and P be linear maps from the (x_1, x_2) plane to the (y_1, y_2) plane:

   L is rotation by 90 degrees counterclockwise.

   M is reflection across the line $x_1 = x_2$.

   $N\vec{v} := -\vec{v}$ for any vector $\vec{v} \in \mathbb{R}^2$.

a) Find matrices representing each of the linear maps L, M, and N.

b) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM.

c) Which pairs of these maps commute?

d) Which of the following identities are correct—and why?

1) $L^2 = N$	2) $N^2 = I$	3) $L^4 = I$	4) $L^5 = L$
5) $M^2 = I$	6) $M^3 = M$	7) $MNM = N$	8) $NMN = L$

8. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees *counterclockwise*).

b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

9. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.

b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.

c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.

e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$.

f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

g) If A is a square matrix, then $\det A = ?$

h) If A is a square matrix, for any given vector $\mathbf{W}$ can one always find at least one solution of $A\mathbf{X} = \mathbf{W}$? Why?

1[Last revised: January 16, 2014]