Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 10

DUE: In class Thursday, Thurs. April 17 *Late papers will be accepted until 1:00 PM Friday.*

For the coming week, please re-read Chapter 8.1-8.2 in the text and the notes:

`http://www.math.upenn.edu/~kazdan/312F12/notes/quadratic/quadratic.pdf` Quadratic Polynomials

`http://www.math.upenn.edu/~kazdan/312F12/notes/max-min-notesJan09/max-min.pdf`

and

`http://www.math.upenn.edu/~kazdan/312S13/notes/MultInt1.pdf` Change of Variable in a Multiple Integral.

1. [BRETSCHER, SEC. 8.1 #13] Consider a symmetric 3×3 matrix A with $A^2 = I$ but $A \ne I$.Is A necessarily the reflection across a subspace of $\mathbb{R}^3$?

2. [BRETSCHER, SEC. 8.2 #18] Sketch the curve of points in the plane that satisfy $9x_1^2 - 4x_1x_2 + 6x_2^2 = 1$.

3. Let $A := \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
   a) Find the eigenvalues and eigenvectors of A.
   b) Find an orthogonal transformation R so that $R^{-1}AR$ is a diagonal matrix.

4. Let $A = (a_{ij})$ be an $n\times n$ symmetric matrix. Show that if the constant c is sufficiently large, then the matrix $B := A + cI$ is positive definite.

5. [BRETSCHER, SEC. 8.2 #56] Let $A = (a_{ij})$ be a real symmetric matrix. Show there is an eigenvalue λ such that $\lambda \ge a_{11}$.

6. Let $\lambda_{\max}$ be the largest eigenvalue of the real symmetric matrix A. Show that
$$\lambda_{\max} = \max \frac{\langle \vec{x},\, A\vec{x}\rangle}{\|\vec{x}\|^2}$$
   for all vectors $\vec{x} \ne 0$.

7. Let C be a real symmetric matrix. If $\langle \vec{x},\, C\vec{x}\rangle = 0$ for all vectors $\vec{x}$, show that $C = 0$.

8. If the symmetric matrix A is positive definite, show that $\det A > 0$. In particular, if $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive definite, then $ac - b^2 > 0$.

9. Let $A = (a_{ij})$ be a positive definite $n \times n$ matrix.

   a) For any integer k, $1 \le k \le n$, show that the 1×1 matrix $C := (a_{kk})$ is positive definite. Why does this show that the diagonal elements of a positive definite matrix must be positive?

   b) For any integers k, ℓ, $1 \le k < \ell \le n$, show that the 2×2 matrix $M := \begin{pmatrix} a_{kk} & a_{k\ell} \\ a_{k\ell} & a_{\ell\ell} \end{pmatrix}$ is positive definite – and hence that $a_{kk}a_{\ell\ell} - a_{k\ell}^2 > 0$.

   c) If $1 \le k \le n$, show that the $k \times k$ sub-matrix consisting only of the first k rows and columns of A is positive definite.

10. a) Let $D := \begin{pmatrix} 4 & 0 \\ 0 & 25 \end{pmatrix}$ Find a positive definite symmetric matrix P so that $P^2 = D$ (we call P the *square root* of D)

    b) Let $A := \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$. Find a positive definite (symmetric) matrix P so that $P^2 = A$.

    c) Show that every positive definite symmetric matrix A has a positive definite square root.

11. Just as we can write and complex number $z \ne 0$ in the polar form $z = re^{i\theta}$, we can write any real invertible matrix A in the form $A = PR$, where P is positive definite and R is an orthogonal matrix, by the following steps:

    a) If, A is invertible, show that AA^* is positive definite and hence, by problem 10c, there is a positive definite matrix P so that $AA^* = P^2$.

    b) Using this P, let $R := P^{-1}A$. Show that $RR^* = I$, and hence that R is an orthogonal matrix. Thus $A = PR$, as desired.

12. [Bretscher, Sec. 8.1 #38] Let A be a symmetric 2×2 matrix with eigenvalues -2 and 3 and $\vec{u} \in \mathbb{R}^2$ any unit vector. What are the possible values of $\langle \vec{u}, A\vec{u} \rangle$? Illustrate your answer in terms of the unit circle and its image under A.

13. Let A be a 3×3 matrix whose eigenvalues are $-1 \pm i$ and -2. If $\vec{x}(t)$ is a solution of $\dfrac{d\vec{x}}{dt} = A\vec{x}$, show that $\lim_{t\to\infty} \vec{x}(t) = 0$ independent of the initial value $\vec{x}(0)$.

14. Let $f(x,y) = (x^2 + 4y^2)e^{1-x^2-y^2}$. Find and classify all of the critical points of f as local maxima, minima, or saddles.

15. Let $A := \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Solve $\dfrac{d^2\vec{x}(t)}{dt^2} + A\vec{x}(t) = 0$ with $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{x}\,'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

    [Remark: If A were the diagonal matrix $\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$, then this problem would have been simple. This problem assumes you know how to solve scalar ordinary differential equations like $u'' + 25u = 0$ and $u'' - 25u = 0$. Review your Math 240 text.]

16. a) Compute $\displaystyle\iint_{R^2} e^{-(x^2+y^2)}\,dx\,dy$. [Hint: polar coordinates].

    b) Compute $\displaystyle\iint_{R^2} e^{-(2x^2+6xy+9y^2)}\,dx\,dy$.

    [See: `http://www.math.upenn.edu/~kazdan/312S13/notes/MultInt1.pdf` Change of Variable in a Multiple Integral.]

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let A be a positive definite $n \times n$ matrix, let E be the "ellipsoid" of points $\vec{x}$ where $\langle \vec{x}, A\vec{x} \rangle = 1$, and for a unit vector $\vec{b}$ let $\mathcal{P}$ be the set of points on the hyperplane $\langle \vec{x}, \vec{b} \rangle = c$. Find the two values of c so that the hyperplane $\mathcal{P}$ is tangent to E (thus, the hyperplane intersects the ellipsoid in exactly one point).

[Hint: This is easy if $A = I$. Use Problem 10c to make an appropriate change of variables that reduces to this case.]

[Last revised: April 12, 2014]