Homework 1 Solutions

1. Let $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$. Compute the inverse of A and of A^2.

   SOLUTION $A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$. Squaring this we find the inverse of A^2 is

$$(A^2)^{-1} = (A^{-1})^2 = \begin{pmatrix} 14 & -25 \\ -5 & 9 \end{pmatrix}.$$

   A much longer computation is to compute A^2 first and then compute it's inverse.

2. Solve all of the following equations. [Note that the left sides of these equations are identical.]

$$\text{a).}\ \begin{aligned} 2x+5y &= 5 \\ x+3y &= -1 \end{aligned} \qquad \text{b).}\ \begin{aligned} 2x+5y &= 0 \\ x+3y &= -2 \end{aligned} \qquad \text{c).}\ \begin{aligned} 2x+5y &=1 \\ x+3y &=0 \end{aligned} \qquad \text{d).}\ \begin{aligned} 2x+5y &=2 \\ x+3y &=1 \end{aligned}$$

   SOLUTION: All of these have the form $A\vec{v} = \vec{b}$ where A is the matrix whose inverse you computed in Problem 1a). Thus $\vec{v} = A^{-1}\vec{b}$ where $\vec{b}$ is the right hand side of each of these equations. The computations are now very short.

3. [Bretscher, Sec.2.1 #13] Finding the inverse of a matrix A means solving the system of equations $A\vec{x} = \vec{y}$ for $\vec{x}$, so $\vec{x} = A^{-1}\vec{y}$.

   a) Let $A := \left(\begin{smallmatrix} 1 & 2 \\ c & 6 \end{smallmatrix}\right)$. With your bare hands (as on page 2 of the textbook – not using anything about determinants) show that A is invertible if and only if $c \neq 3$.

   SOLUTION This asks for which c you can solve the equations

$$\begin{aligned} x_1 + 2x_2 &= y_1 \\ cx_1 + 6x_2 &= y_2. \end{aligned}$$

   Multiply the first equation by 3 and subtract it from the second equation. The resulting equation $(c-3)x_1 = y_2 - 3y_1$ can always be solved unless $c = 3$.

   b) Let $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. With your bare hands (not using anything about determinants) show that M is invertible if and only if $ad - bc \neq 0$.

   SOLUTION Here the equations to be solved are

$$\begin{aligned} ax_1 + bx_2 &= y_1 \\ cx_1 + dx_2 &= y_2 \end{aligned}.$$

   Multiply the first equation by d and the second by b and subtract to find

$$(ad - bc)x_1 = dy_1 - by_2.$$

If $ad-bc \neq 0$ one can solve this for x_1. Note that $ad-bc \neq 0$ implies that b and d cannot both be zero so we can now use the value of x_1 to solve for x_2 use that to find x_2.

If $ad-bc = 0$ then $dy_1 - by_2 = 0$. This condition on y_1 and y_2 means the map defined by A is not onto so the matrix is not invertible.

4. Let A and B be 2×2 matrices.

   a) If B is invertible and $AB = 0$, show that $A = 0$.

      SOLUTION: Multiply the equation on the right by B^{-1} to get $A = ABB^{-1} = 0$.

   b) Give and example where $AB = 0$ but $BA \neq 0$.

      SOLUTION Let $A := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

   c) Find an example of a 2×2 matrix with the property that $A^2 = 0$ but $A \neq 0$.

      SOLUTION Let $A := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ or $A := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

   d) Find all invertible $n \times n$ matrices A with the property $A^2 = 3A$.

      SOLUTION Multiply both sides by A^{-1} and obtain $A = 3I_n$.

5. [Bretscher, Sec.2.3 #19] Find all the matrices that commute with $A := \left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$.

   SOLUTION By a straightforward computation, these matrices all have the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) = aI + bA$ for any scalars a and b. [Note that for *any* square matrix M the matrices $aI + bM$ always commute with M. In this case, these are the only matrices that do so.]

6. a) Find a real 2×2 matrix A (other than $A = \pm I$) such that $A^2 = I$.

      SOLUTION A reflection, say across the vertical axis: $A := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$

   b) Find a real 2×2 matrix A [other than $A = \pm I$] such that $A^4 = I$ but $A^2 \neq I$.

      SOLUTION A rotation by 90 degrees ($\pi/2$). $A := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

7. Let L, M, and P be linear maps from the (two dimensional) plane to the plane:

   L is rotation by 90 degrees counterclockwise.

   M is reflection across the vertical axis

   $Nv := -v$ for any vector $v \in \mathbb{R}^2$ (reflection across the origin)

   a) Find matrices representing each of the linear maps L, M, and N.

      SOLUTION

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N = -I.$$

b) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM. NOTE; LM is the map $LM(\vec{x}) := L(M\vec{x})$ means you first apply M to $\vec{x}$ and then apply L to the result. Thus LM means first reflect across the vertical axis and then rotate by 90 degrees counterclockwise.

c) Which pairs of these maps commute?

SOLUTION $ML \neq LM$ and N commutes with every 2×2 matrix.

d) Which of the following identities are correct—and why?

1)	$L^2 = N$	2)	$N^2 = I$	3)	$L^4 = I$	4)	$L^5 = L$
5)	$M^2 = I$	6)	$M^3 = M$	7)	$MNM = N$	8)	$NMN = L$

SOLUTION All are correct *except* #8. Since $N = -I$, then $NMN = M \neq L$.

8. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees *counterclockwise*).

b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

SOLUTION You can use the Theorem 2.2.3 in Bretscher on p.67 to find the matrix for any rotation. A reflection across the horizontal axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

9. Let A be a matrix, not necessarily square. Say $\vec{v}$ and $\vec{w}$ are particular solutions of the equations $A\vec{v} = \vec{y}_1$ and $A\vec{w} = \vec{y}_2$, respectively, while $\vec{z} \neq 0$ is a solution of the homogeneous equation $A\vec{z} = 0$. Answer the following in terms of $\vec{v}$, $\vec{w}$, and $\vec{z}$.

a) Find some solution of $A\vec{x} = 3\vec{y}_1$. SOLUTION: $3\vec{v}$

b) Find some solution of $A\vec{x} = -5\vec{y}_2$. SOLUTION: $-5\vec{w}$

c) Find some solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$. SOLUTION: $3\vec{v} - 5\vec{w}$

d) Find another solution (other than $\vec{z}$ and 0) of the homogeneous equation $A\vec{x} = 0$. SOLUTION: $7\vec{z}$

e) Find *two* solutions of $A\vec{x} = \vec{y}_1$. SOLUTION: $\vec{v}$ and $\vec{v} + 7\vec{z}$

f) Find another solution of $A\vec{x} = 3\vec{y}_1 - 5\vec{y}_2$. SOLUTION: $3\vec{v} - 5\vec{w} + 7\vec{z}$

g) If A is a square matrix, then $detA =$? SOLUTION: $\det A = 0$ since $A\vec{z} = 0$ with $\vec{z} \neq 0$.

h) If A is a square matrix, for any given vector $\vec{w}$ can one always find at least one solution of $A\vec{x} = \vec{w}$? Why?

SOLUTION: No. Since the kernel of A contains $\vec{z} \neq 0$ so it is not just 0, Since A is a square matrix, it is thus not onto. [Note that a non-square matrix A, say $A : \mathbb{R}^3 \to \mathbb{R}^2$ always has a non-trivial kernel but it can be onto,]