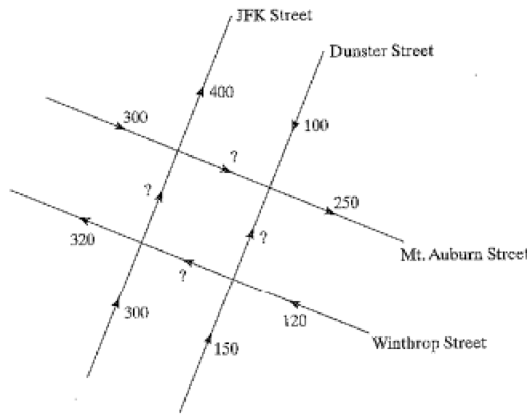
Homework 2 Solutions

1. [Bretscher, Sec. 1.2 #44] The sketch represents a maze of one-way streets in a city. The traffic volume through certain blocks during an hour has been measured. Suppose that the number of vehicles leaving this area during this hour was exactly the same as the number of vehicles entering it.

   What can you say about the traffic volume at the four locations indicated by question marks? Can you determine exactly how much traffic there was on each block? If not, find the highest and lowest possible traffic volumes.

   Solution At each of the four intersection points created we know that the sum of incoming traffic must equal the outgoing traffic.

   Notation: Let w, x, y, and z be the traffic at the question marks beginning at JFK Street and going counter-clockwise. Then we obtain directly four equations:

$$\begin{aligned} x+y &= 270 \\ y+z+100 &= 250 \\ 300+w &= 400+z \\ x+300 &= 320+w, \end{aligned} \quad \text{that is,} \quad \begin{aligned} x+y &= 270 \\ y+z &= 150 \\ w-z &= 100 \\ x-w &= 20 \end{aligned}$$

   It's easy to see that the system has infinitely many solutions since any three equations determines the fourth one. Also we can easily determine the possible values of x, y, z, w since all are non-negative variables. We thus obtain from the second equation that that $0 \le y,\ z \le 150$, with both extrema values possible, Since $y \le 150$, from the first equation $120 \le x \le 270$. Again, both extreme values can occur. The third and fourth equations now show that $100 \le w \le 250$ and again both extreme values can occur.

2. Consider the system of equations

$$\begin{aligned} x+y-\ \ z &= a \\ x-y+2z &= b \\ 3x+y \qquad &= c \end{aligned}$$

   a) Find the general solution of the homogeneous equation.

   b) If $a=1$, $b=2$, and $c=4$, then a particular solution of the inhomogeneous equations is $x=1$, $y=1$, $z=1$. Find the most general solution of these inhomogeneous equations.

c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have *no* solution.

d) If you view these equations as defining a linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$, find a basis for $\ker(A)$ and for image (A).

SOLUTION For part (a) by adding the first two equations we get $z = -2x$ and the third one gives $y = -3x$. Hence the general solution of the homogeneous equation is $(x, y, z) = (t, -3t, -2t)$ for any $t \in \mathbb{R}$.

For part (b) the general solution of the inhomegeneous equations is a particular solution plus the general solution of the homogeneous equations, so $(x, y, z, w) = (1, 1, 1) + (t, -3t, -2t) = (1 + t, 1 - 3t, 1 - 2t)$.

For part (c), two times the first equation plus the second one will give $3x + y = 4$ which contradicts with the third equation.

For part (d), the general solution of the homogeneous equations give us directly a basis for the kernel of A, namely $\{(1, -3, -2\}$ since $\ker(A) = \{(t, -3t, -2t) | t \in \mathbb{R}\}$.

Since $\dim(\ker(A)) = 1$, we know that $\dim(\mathrm{im}(A)) = 2$. Also. thes image is spanned by the columns of A. It is clear that any two of the columns of A are linearly independent so we can use any two of the columns of A as a basis of the image of A.

3. Let A and B both be $n \times n$ matrices. What's wrong with the formula $(A + B)^2 = A^2 + 2AB + B^2$? Prove that if this formula is valid for A and B, then A and B commute.

   SOLUTION Since $(A + B)^2 = A^2 + AB + BA + B^2$. he formula is correctly only when $AB + BA = 2AB$, that is, $AB = BA$.

4. [Bretscher, Sec.2.2 #17] Let $A := \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$, where $a^2 + b^2 = 1$. Find two perpendicular non-zero vectors $\vec{v}$ and $\vec{w}$ so that $A\vec{v} = \vec{v}$ and $A\vec{w} = -\vec{w}$ (write the entries of $\vec{v}$ and $\vec{w}$ in terms of a and b). Conclude that thinking of A as a linear map it is an orthogonal reflection across the line $\mathcal{L}$ spanned by $\vec{v}$.

   SOLUTION The equation $A\vec{v} = \vec{v}$ means $(A - I)\vec{v} = 0$ so we solve these homogeneous equations:

$$\begin{aligned} (a-1)v_1 + bv_2 &= 0 \\ bv_1 - (a+1)v_2 &= 0 \end{aligned}$$

Case 1. $b \neq 0$ so $v_1 = [(a + 1)/b]\, v_2$, that is, $\vec{v} = \begin{pmatrix} (a+1)/b \\ 1 \end{pmatrix} v_2$ for any scalar v_2. It is simplest to pick $v_2 = b$ since then $\vec{v} = \begin{pmatrix} a+1 \\ b \end{pmatrix}$.

Case 2. $b = 0$ so $a = \pm 1$. If $a = 1$ then $\vec{v} = (v_1, 0)$. If $a = -1$, then $\vec{v} = (0, v_2)$

The computation for $\vec{w}$ is similar.

5. [Bretscher, Sec.2.2 #31] Find a nonzero 3×3 matrix A so that $A\vec{x}$ is perpendicular to $\vec{v} := (1,2,3)$ for all vectors $\vec{x} \in \mathbb{R}^3$.

SOLUTION We want $\vec{v} \cdot A\vec{x} = 0$ for all x. As a computation this is straightforward: we want

$$\begin{aligned} 0 = \vec{v} \cdot A\vec{x} =& 1[a_{11}x_1 + a_{12}x_2 + a_{13}x_3] \\ &+ 2[a_{21}x_1 + a_{22}x_2 + a_{23}x_3] \\ &+ 3[a_{31}x_1 + a_{32}x_2 + a_{33}x_3] \\ =& (a_{11} + 2a_{21} + 3a_{31})x_1 \\ &+ (a_{12} + 2a_{22} + 3a_{32})x_2 \\ &+ (a_{13} + 2a_{23} + 3a_{33})x_3 \end{aligned}$$

for all x_1, x_2, and x_3. This means the coefficients of x_1, x_2, and x_3 must all be zero. One way to get this is to pick a_{11}, a_{21}, and a_{31} so that $a_{11} + 2a_{21} + 3a_{31} = 0$, say $a_{11} = 2$, $a_{21} = -1$, $a_{31} = 0$ and let all the other elements of A be zero.

More conceptually, if we name the three columns of A as A_1, A_2, and A_3, then notice that $A\vec{x} = A_1x_1 + A_2x_2 + A_3x_3$ (the image of A is all possible linear combinations of the columns of A). Thus we want the 3 columns of A to be orthogonal to $\vec{v}$, so we can pick A_1 to be orthogonal to $\vec{v}$ (say $a_{11} = 2$, $a_{21} = -1$, $a_{31} = 0$) and simply have $A_2 = A_3 = 0$.

6. [Bretscher, Sec.2.3 #48]

   a) If $A := \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, compute AB and A^{10}.

   b) Find a 2×2 matrix A so that $A^{10} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

   SOLUTION For (a): $AB = \left(\begin{smallmatrix} 1 & a+b \\ 0 & 1 \end{smallmatrix}\right)$ so $A^{10} = \left(\begin{smallmatrix} 1 & 10a \\ 0 & 1 \end{smallmatrix}\right)$.

   For (b): $A = \left(\begin{smallmatrix} 1 & \frac{1}{10} \\ 0 & 1 \end{smallmatrix}\right)$

7. Which of the following subsets of $\mathbb{R}^2$ are actually linear subspaces? Explain.

   a) $\{(x,y) \mid xy = 0\}$

   b) $\{(x,y) \mid x \text{ and } y \text{ are both integers}\}$

   c) $\{(x,y) \mid x + y = 0\}$

   d) $\{(x,y) \mid x + y = 2\}$

   e) $\{(x,y) \mid x + y \geq 0\}$

   SOLUTION (a): No since $(1,0) + (0,1) = (1,1)$ doesn't belong to the set.

   (b): No since $0.5(1,0)$ doesn't belong to the set.

   (c): Yes.

   (d): No since (0,0) doesn't belong to the set.

   (e): No since $-(1,0)$ doesn't belong to the set.

8. Which of the following sets are linear spaces? Why?

   a) $\{\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0\}$

   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.

   c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\, dx = 0$.

   d) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$ (you are *not* being asked to actually find these solutions).

   e) The set of all 2×3 matrices with real coefficients?

   f) The set of all 2×2 invertible real matrices?

   SOLUTION All of these *except* (f) are linear spaces. The set of invertible matrices does not, for instance, contain the zero matrix.

9. Proof or counterexample. Here L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

   a) If L is invertible, then L^{-1} is also invertible.

      SOLUTION The inverse of L^{-1} is just L.

   b) If $L\vec{v} = 5\vec{v}$ for all vectors $\vec{v}$, then $L^{-1}\vec{w} = (1/5)\vec{w}$ for all vectors $\vec{w}$.

      SOLUTION True since $L = 5I$ hence $L^{-1} = \frac{1}{5}I$.

   c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

      SOLUTION True. A geometric approach makes this easy to verify.

   d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

      SOLUTION True. A rotation by 315 degrees is the same as rotating by -45 degrees.

   e) The zero map ($0\vec{v} := 0$ for all vectors $\vec{v}$) is invertible.

      SOLUTION False. The zero map is not one-to-one since it maps all vectors to the origin.

   f) The identity map ($I\vec{v} := \vec{v}$ for all vectors $\vec{v}$) is invertible.

      SOLUTION True since it's inverse is itself

   g) If L is invertible, then $L^{-1}0 = 0$.

      SOLUTION True, since L^{-1} is a linear map and this holds for any linear map.

   h) If $L\vec{v} = 0$ for some non-zero vector $\vec{v}$, then L is not invertible.

      SOLUTION True since then L is not one-to-one.

   i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

      SOLUTION False. Reflections, say across the horizontal axes, also have this property.

10. a) Assume the kernel of T is trivial, that is, the only solution of the homogeneous equation $T\vec{x}=0$ is $\vec{x}=0$. Prove that if $T(\vec{x})=T(\vec{y})$, then $\vec{x}=\vec{y}$.

    b) Conversely, if T has the property that "if $T(\vec{x})=T(\vec{y})$, then $\vec{x}=\vec{y}$," show that the kernel of T is trivial.

    SOLUTION (a) Suppose $T(\vec{x})=T(\vec{y})$, then $T(\vec{x}-\vec{y})=0$ so $\vec{x}-\vec{y}$ lies in the kernel; hence $\vec{x}-\vec{y}=0$.

    (b): Suppose z is in the kernel namely $T(z)=0$. Since $T(0)=0$ for linear maps we get that $z=0$.

11. Say $\vec{v}_1,\dots,\ \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$ and $T:\mathbb{R}^n\to\mathbb{R}^n$ is a linear map.

    a) Show by an example, say for $n=2$, that $T\vec{v}_1,\dots,T\vec{v}_n$ need not be linearly independent.

    b) However, show that if the kernel of T is trivial, then these vectors $T\vec{v}_1,\dots,T\vec{v}_n$ are linearly independent.

    SOLUTION For (a): $(1,1),(1,-1)$ are linearly independent but if T is the projection on x-axis their images are not.

    For (b): Suppose $\sum a_iT\vec{v}_i=0$, then $T(\sum a_i\vec{v}_i)=0$ so $\sum a_i\vec{v}_i=0$. Since the $\vec{v}_i$ are linearly independent we get $a_i=0$ for all i.

12. [LIKE BRETSCHER, SEC. 2.4 #40].

    a) If A has two equal rows show that it is not onto (and hence not invertible).

    SOLUTION: In thinking about inverses I always prefer to think in terms about solving the equation $A\vec{x}=\vec{b}$. The left side of each equation corresponds to one of the rows of A, so the left side of two of the equations are identical. Consequently there is no solution except in the rare case that their right sides of these two rows are identical. Thus A is not onto and hence not invertible.

    In greater detail, say the first two rows are identical. Then the first two equations of $A\vec{x}=\vec{b}$ are

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n &= b_1 \\ a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n &= b_2 \end{aligned}.$$

    Since the left-hand sides of these are identical, these cannot have a solution unless $b_1=b_2$. Consequently, the equation $A\vec{x}=\vec{b}$ do not have a solution except for very restricted vectors $\vec{b}$.

    b) If a matrix has two equal columns, show that it is not one-to-one and hence not invertible.

    SOLUTION: Say the columns of A are the vectors $A_1,\ A_2,\dots,A_n$ Then the homogeneous equation $A\vec{x}=\vec{0}$ is $A_1x_1+A_2x_2+\cdots+A_nx_n=\vec{0}$. Say the first two columns are equal, $A_1 = A_2$. Then the homogeneous equation is $A_1(x_1 + x_2) + A_3x_3 + \cdots + A_nx_n = \vec{0}$. Clearly, any vector of the form $\vec{x} = (c, -c, 0, 0, \ldots, 0)$ is a solution for any constant c. Since the homogeneous equation has a solution other than $\vec{0}$, the kernel of A is not zero so A is not one-to-one and thus cannot be invertible.

REMARK: Neither part of this used that A is a square matrix.

13. Let V be the linear space of smooth real-valued functions and $L : V \to V$ the linear map defined by $Lu := u'' + u$.

   a) Compute $L(e^{2x})$ and $L(x)$.

   SOLUTION $L(e^{2x}) = 4e^{2x} + e^{2x} = 5e^{2x}$. $\qquad L(x) = 0 + x = x$.

   b) Find particular solutions of the inhomogeneous equations

$$i).\ u'' + u = 7e^{2x}, \qquad ii).\ w'' + w = 4x, \qquad iii).\ z'' + z = 7e^{2x} - 3x$$

   SOLUTION i). $u_{\text{part}}(x) = \frac{7}{5}e^{2x}$. ii). $w_{\text{part}}(x) = 4x$. iii). $z_{\text{part}}(x) = \frac{7}{5}e^{2x} - 3x$.

14. Let $A : \mathbb{R}^3 \to \mathbb{R}^5$ and $B : \mathbb{R}^5 \to \mathbb{R}^2$.

   a) What are the maximum and minimum values for the dimension of the kernels of A, B, and BA?

   b) What are the maximum and minimum values for the dimension of the images of A, B, and BA?

   SOLUTION $0 \le \dim(\ker(A)) \le 3$, $\quad 0 \le \dim(\operatorname{im}(A)) \le 3$,

   $3 \le \dim(\ker(B)) \le 5$, $\quad 0 \le \dim(\operatorname{im}(B)) \le 2$,

   $1 \le \dim(\ker(BA)) \le 3$, $\quad 0 \le \dim(\operatorname{im}(BA)) \le 2$.

15. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another. If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

   SOLUTION For this problem it is simplest to think of a straight line as the position of a particle at time t with constant velocity $\vec{v}$, thus $\vec{x}(t) = \vec{v}t + \vec{x}_0$, where $\vec{x}_0$ is its position at $t = 0$. Then $\vec{v}$ determines the slope of the line. Another line $\vec{w}(t) = \vec{u}t + \vec{w}_0$ is parallel to this one only if $\vec{u} = c\vec{v}$ for some constant c.

   The image of the straight line is $A\vec{x}(t) = (A\vec{v})t + A\vec{x}_0$. these lines all have the same velocity vector, $A\vec{v}$ independent of the point $\vec{x}_0$ and are therefore parallel.

16. In $\mathbb{R}^n$ let $\vec{e}_1 = (1, 0, 0, \ldots, 0)$, $\vec{e}_2 = (0, 1, 0, \ldots, 0)$ and let $\vec{v}$ and $\vec{w}$ be any non-zero vectors in $\mathbb{R}^n$.

a) Find an invertible matrix A with $A\vec{e}_1 = \vec{e}_2$

SOLUTION: Let the first column of A be the vector $\vec{e}_2$ and for the remaining columns of B use $\vec{e}_1, \vec{e}_3, \ldots, \vec{e}_n$ in any order.

b) Show there is an invertible matrix B with $B\vec{e}_1 = \vec{v}$.

SOLUTION: Let Let the first column of B be the vector $\vec{v}$. Since $\vec{v} \neq 0$, some component, say v_j of $\vec{v}$ is not zero for the remaining columns use any vectors that extend $\vec{v}$ to a basis for $\mathbb{R}^n$. For instance, if the first component of $\vec{v}$ is not zero, you can use the standard basis vectors $\vec{e}_1, \ldots, \vec{e}_n$ *except* $\vec{e}_j$ (in any order) for the remaining columns of B.

c) Show there is an invertible matrix M with $M\vec{w} = \vec{v}$.

SOLUTION: As in the previous part, let A be an invertible matrix that maps $\vec{e}_1$ to $\vec{w}$. Then let $M := BA^{-1}$.

Note: I wrote all vectors as row vectors instead of column vectors. You can consider them as if they were column vectors if you get confused.