Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 3

Due: In class Thursday, Thurs. Feb. 6. *Late papers will be accepted until 1:00 PM Friday.*

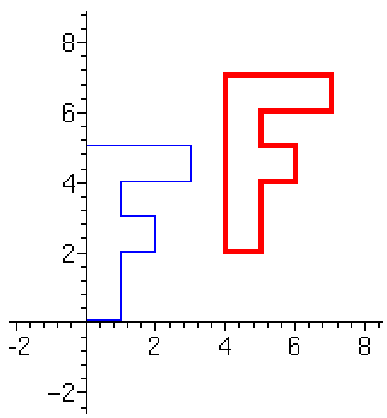
For the coming week, please review Sections 3.1, 3.2, and 3.3 in the textand real Section 3.4 (lightly) and read Chapter 4 Sections 4.1 and 4.2. We have already covered most of Chapter 4.
Later we will return in greater detail to the material in Section 3.4 on Coordinates.

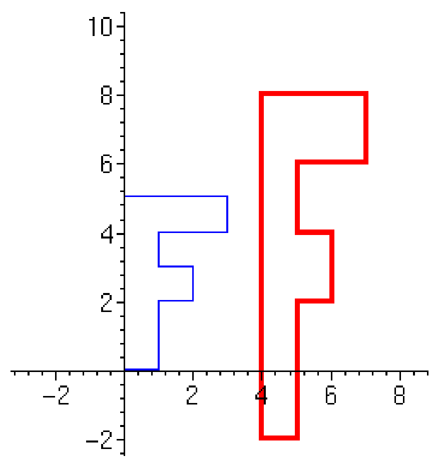
1. Let A, B, and C be $n \times n$ matrices with A and C invertible. Solve the equation $ABC = I - A$ for B.

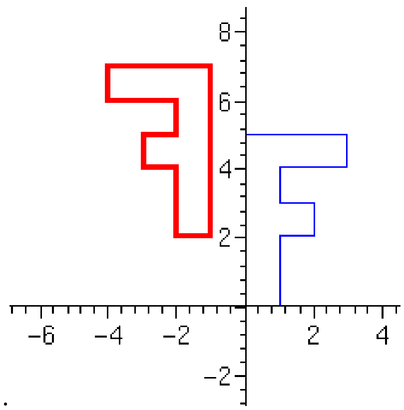
2. If a square matrix M has the property that $M^4 - M^2 + 2M - I = 0$, show that M is invertible. [Suggestion: . Find a matrix N so that $MN = I$. This is very short.]

3. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,
$$F(X) = V + AX,$$
where V is a vector. Note that $F(0) = V$.

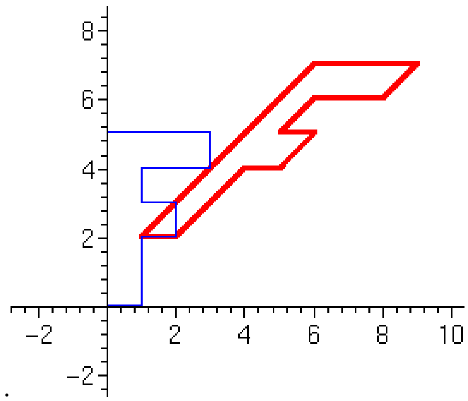
   Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].

a). 

b). 



c).



d).

4. a). Use Theorems from Section 3.3 (or from class) to explain the following carefully.
   a) If V and W are subspaces with V contained inside of W, why is $\dim V \leq \dim W$?
   b) If $\dim V = \dim W$, explain why $V = W$.

5. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$ and $AB : \mathbb{R}^2 \to \mathbb{R}^2$.
   a) Why must there be a non-zero vector $\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = 0$.
   b) Show that the 3×3 matrix BA can *not* be invertible.
   c) Give an example showing that the 2×2 matrix AB might be invertible.

6. Let A be a square matrix. If A^2 is invertible, show that A is invertible. [NOTE: You cannot use the formula $(AB)^{-1} = B^{-1}A^{-1}$ because it presumes you *already* know that both A and B are invertible. For non-square matrices, it is possible for AB to be invertible while neither A nor B are (see the last part of the previous problem).]

7. [BRETSCHER, SEC. 2.4 #35] An $n \times n$ matrix A is called *upper triangular* if all the elements below the *main diagonal*, a_{11} a_{22}, $\ldots a_{nn}$ are zero, that is, if $i > j$ then $a_{ij} = 0$.
   a) Let A be the upper triangular matrix
   $$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$
   For which values of a, b, c, d, e, f is A invertible? HINT: Write out the equations $A\vec{x} = \vec{y}$ explicitly.
   b) If A is invertible, is its inverse also upper triangular?
   c) Show that the product of two $n \times n$ upper triangular matrices is also upper triangular.

d) Show that an upper triangular $n \times n$ matrix is invertible if none of the elements on the main diagonal are zero.

e) Conversely, if an upper triangular $n \times n$ matrix is invertible show that none of the elements on the main diagonal can be zero.

8. [SEE BRETSCHER, SEC. 3.2 #6] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and let $W = U \cap V$. Find all possible values for the dimension of W.

9. [SEE BRETSCHER, SEC. 3.2 #50] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.

   a) Show that W is a linear space.

   b) Find all possible values for the dimension of W.

10. Say you have k linear algebraic equations in n variables; in matrix form we write $A\vec{x} = \vec{y}$. Give a proof or counterexample for each of the following.

    a) If $n = k$ there is always *at most one* solution.

    b) If $n > k$ you can *always* solve $A\vec{x} = \vec{y}$.

    c) If $n > k$ the nullspace (= kernel) of A has dimension greater than zero.

    d) If $n < k$ then for *some* $\vec{y}$ there is *no* solution of $A\vec{x} = \vec{y}$.

    e) If $n < k$ the *only* solution of $A\vec{x} = 0$ is $\vec{x} = 0$.

11. [BRETSCHER, SEC. 3.3 #30] Find a basis for the subspace of $\mathbb{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let $L\colon V \to V$ be a linear map on a linear space V.

a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^j \subset \ker L^{j+1}$ for all $j \geq 1$, so the kernel of L^j can only get larger as j increases.

b) If $\ker L^j = \ker L^{j+1}$ for some integer j, show that $\ker L^k = \ker L^{k+1}$ for all $k \geq j$. [HINT: $L^{k+2} = L^{k+1}L$.]

MORAL: If at some step the kernel of L^j does not get larger, then it never gets larger for any $k > j$.

c) Let A be an $n \times n$ matrix. If $A^k = 0$ for some integer k we say that A is *nilpotent*. If $A : \mathbb{R}^7 \to \mathbb{R}^7$ is nilpotent, show that $A^7 = 0$. [HINT: What is the largest possible value of $\dim(\ker A)$?]

Generaliz this to a nilpotent $A : \mathbb{R}^n \to \mathbb{R}^n$.

[Last revised: June 1, 2014]