Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 4 Solutions

DUE: In class Thursday, Thurs. Feb. 13. *Late papers will be accepted until 1:00 PM Friday.*

Reminder: Exam 1 is on Tuesday, Feb. 18, 9:00–10:20. No books or calculators but you may always use one 3" × 5" card with handwritten notes on both sides.

For the coming week, please review Chapter 4 Sections 4.1 and 4.2. Also read Sections 5.1 and 5.2 (we will skip the QR Factorization) and the notes
`http://www.math.upenn.edu/~kazdan/312S13/notes/vectors/vectors10.pdf` on Vectors and Least Squares and
`http://www.math.upenn.edu/~kazdan/312S13/notes/OrthogProj.pdf` on Orthogonal Projections.
Later we will return in greater detail to the material in Sections 3.4 and 4.3.

1. Find a basis for the linear space of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the property that $a+d=0$. What is the dimension of this space?

   SOLUTION Such matrices are of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ hence any element of this linear space can be written as

   $$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

   Hence the three matrices appearing in this decomposion form a basis of this linear space and the dimension is 3.

2. Find a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is exactly the plane

   $$\{(x_1,\, x_2,\, x_3) \subset \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\}.$$

   SOLUTION Let $L = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3. a) [LIKE BRETSCHER, SEC. 4.2 #66] Find the kernel of the map $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by $T(u) := u' - 4u$. What is the dimension of the kernel?

   b) Repeat this for $Tu := u'' - 4u$.

   SOLUTION

   a) The kernel of T will be the set of solutions of $u' = 4u$ which is $\{ce^{4x} | c \in \mathbb{R}\}$. So it has dimension 1.

b) The kernel of T will be the set of solutions of $u'' = 4u$ which is $\{ae^{2x} + be^{-2x} | a. b \in \mathbb{R}\}$. So the kernel has dimension 2.

4. We want to approximately compute $\int_0^2 \frac{1}{1+x^2}\,dx$ by partitioning the interval $0 \le x \le 2$ into four sub-intervals whose end point are $x_0 = 0, x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$. of width $h = x_{i+1} - x_i = 1/2$. In each sub-interval replace the integrand by a simpler function.

TRAPEZOIDAL RULE: Approximate the function $f(x)$ in each sub-interval $[x_i, x_{i+1}]$ by a straigh line joining its end points: $(x_1,\ f(x_i))$, $(x_{i+1},\ f(x_{i+1}))$.

SOLUTION The values of the function $y_i = f(x_i) = \frac{1}{1+x_i^2}$ at the points x_i are

$$y_0 = 1,\ y_1 = 4/5 = 0.8,\ y_2 = 1/2 = 0.5\ , y_3 = 4/13 = 0.3077, y_4 = 1/5 = 0.2 \quad (1)$$

Then the straight line between $(x_i, y_i), (x_{i+1}, y_{i+1})$ is given by

$$y = y_i + \frac{y_{i+1} - y_i}{h}(x - x_i).$$

After a simple calculation

$$\int_{x_i}^{x_{i+1}} \left[y_i + \frac{y_{i+1} - y_i}{h}(x - x_i) \right] dx = \frac{h}{2}(y_i + y_{i+1}).$$

Hence the approximation is

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx \sum_{i=0}^{4} \frac{h}{2}(y_i + y_{i+1}) = \frac{1}{4}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) = 1.10384,$$

SIMPSON'S RULE: This works with two sub-intervals at a time, say $x_0 \le x \le x_1$ and $x_1 \le x \le x_2$ and uses a parabola,

$$p(x) := a + bx + cx^2$$

that passes through the three points $(x_0,\ y_0)$, $(x_1,\ y_1)$, and $(x_2,\ y_2)$. The idea is to approximate the area under the function in the interval $x_0 \le x \le x_2$ by the area under the parabola.

SOLUTION The computation is simpler if for the moment we let $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then the conditions that $p(-h) = y_0$, $p(0) = y_1$ and $p(h) = y_2$ become

$$a - bh + ch^2 = y_0, \qquad a = y_1, \qquad \text{and} \qquad a + bh + ch^2 = y_2. \qquad (2)$$

Instead of computing a, b, and c immediately, we first do the next step of the approximate integration using $f(x) \approx p(x)$ in the interval $-h \le x \le h$. Then

$$\int_{-h}^{h} f(x)\,dx \approx \int_{-h}^{h} (a + bx + cx^2)\,dx = 2ah + \frac{2}{3}ch^3 = \frac{h}{3}(6a + 2ch^2).$$

But from equations (2), $a = y_1$ and $2ch^2 = y_0 + y_2 - 2y_1$. Thus

$$\int_{-h}^{h} f(x)\,dx \approx \frac{h}{3}(6y_1 + y_0 + y_2 - 2y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Note that this formula depends only on the spacing, h of the x_j and the values of the corresponding y_j. In particular, we get

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly,

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{h}{3}(y_2 + 4y_3 + y_4).$$

Adding these we obtain

$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4). \tag{3}$$

This is *Simpson's Rule*. More generally, Simpson's Rule works with a pair of adjacent subintervals so there must be an even number of subintervals whose endpoints are x_0, $x_1, \ldots, x_n$. Then it gives

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

We now apply Simpson's Rule (3) to our particular problem (the data (1) and find

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx 1.10513.$$

5. Find a basis for the space $\mathcal{P}_4$ of polynomials $p(x)$ degree at most 4 with the properties $p(1) = 0$ and $p(3) = 0$. What is the dimension of this space?

   Solution We write our polynomials as $p(x) = a_0 + a_1x + a_2x^2 + a_34x^3 + a_4x^4$. Then the conditions $p(1) = 0$ and $p(3) = 0$ are

$$a_0 + a_1 + a_2 + a_3 + a_4 = 0, \qquad \text{and} \qquad a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 = 0.$$

   We use these to solve for a_0 and a_1 in terms of a_2, a_3, and a_4 and then use these in $p(x)$ to obtain

$$p(x) = (3 - 4x + x^2)a_2 + (12 - 13x + x^3)a_3 + (39 - 40x + x^4)a_4.$$

   The three polynomials

$$p_1(x) = 3 - 4x + x^2, \qquad p_2(x) = 12 - 13x + x^3, \qquad \text{and} \qquad p_3(x) = 39 - 40x + x^4$$

are a basis for this space so its dimension is 3.

ALTERNATE The above computation was unpleasant, and leads us to seek a different basis that is better adapted to this space. It uses the observation that each polynomial $p(x)$ in this space must have the quadratic polynomial $(x-1)(x-3)$ as a factor. Thus $p(x)$ must have the form

$$p(x) = (x-1)(x-3)(\alpha + \beta x + \gamma x^2)$$

for any numbers α, β, and γ. This alternate basis consists of the three polynomials

$$q_1(x) = (x-1)(x-3), \qquad q_2(x) = (x-1)(x-3)x, \qquad \text{and} \qquad q_3(x) = (x-1)(x-3)x^2.$$

6. In class we considered the interpolation problem of finding a polynomial of degree n passing through $n+1$ specified distinct points in the plane. To be definite, take $n = 3$, and say our points are (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). This problem involves $\mathcal{P}_3$, and so we could work in the usual basis $\{1, x, x^2, x^3\}$. However, it is easier to use the *Lagrange basis*. The point of this problem is to see vividly why choosing a basis adapted to the problem may involve much less work.

   a) Setup the linear equations you would need to solve to find the polynomial of degree 3 passing through the points $(0, -3)$, $(1, -1)$, $(2, 11)$, and $(-1, -7)$ if you use the usual basis $\{1, x, x^2, x^3\}$. But don't take time to solve these.

   SOLUTION Let $p(x) = a + bx + cx^2 + dx^3$. Then we want:

   $$a = -3, \qquad a+b+c+d = -1, \qquad a+2b+4c+8d = 11, \qquad a-b+c-d = -7,$$

   or equivalently,

   $$b+c+d = 2, \qquad 2b+4c+8d = 14, \qquad -b+c-d = -4,$$

   so we have a 3×3 system to determine the remaining coeffiients b, c, d. Not fun.

   b) Solve the same problem explicitly using the Lagrange basis.

   SOLUTION The Lagrange basis consists of four polynomials $p_i(x)$, $i = 0, \ldots, 3$ with the property that

   $$p_i(x_j) \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

   For instance,

   $$p_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-2)(x+1)}{(0-1)(0-2)(0+1)}.$$

   Then

   $$\begin{aligned} p(x) = &\ y_0\, p_0(x) + y_1\, p_1(x) + y_2\, p_2(x) + y_3\, p_3(x) \\ = &- 3p_0(x) - 1p_1(x) + 11p_2(x) - 7p_3(x) \end{aligned}$$

7. [[Bretscher, Sec. 4.2 #70] Does there exist a polynomial $f(t)$ of degree at most 4 such that $f(2) = 3$, $f(3) = 5$, $f(5) = 7$, $f(7) = 11$, and $f(11) = 2$? If so, how many such polynomials are there? [: Note: This problem only asks if such a polynomial exists. It is not asking you to find it.]

Solution Polynomial Interpolation suggests that there exists a polynomial $f(t)$ of degree at most 4 passing from 5 specified distinct points and using the Lagrange basis as in problem 6 part (b) we can define $f(t)$. Now suppose $f(t), g(t)$ are such polynomials. Then $r(t) = f(t) - g(t)$ is a polynomial of degree at most 4 which has 5 roots. That is a contradiction since a polynomial of degree n has at most n roots. Hence there is a unique such polynomial.

More formally, define the linear map $L : \mathcal{P}_4 \to \mathbb{R}^5$ by the rule

$$Lf := (f(2),\, f(3),\, f(5),\, f(7), f(11))$$

The kernel of L are the quartic polynomial that is zero at the four points $x = 2, 3, 5, 7, 11$. But the only quartic polynomial that has four zeroes is the zero polynomial. Thus $\ker(L) = 0$. Since $\mathcal{P}_4$ and $\mathbb{R}^5$ both have the same dimension, 5, by the rank-nullity theorem L is invertible. Thus given any numbers y_0, y_1, y_2, y_3 and y_4 there is a unique cubic polynomial $f(t)$ so that $f(2) = y_0$, $f(3) = y_1$, $f(5) = y_2$, $f(7) = y_3$, and $f(11) = y_4$.

8. Let $\mathcal{P}_2$ be the linear space of polynomials of degree at most 2 and $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the transformation

$$(T(p))(t) = \frac{1}{t}\int_0^t p(s)\, ds.$$

For instance, if $p(t) = 2 + 3t^2$, then $T(p) = 2 + t^2$.

a) Prove that T is a linear transformation.

Solution Linearity of integral give us directly that T is a linear transformation. Actually if we let $p(s) = a + bs + cs^2$ then $(T(p))(t) = a + \frac{b}{2}t + \frac{c}{3}t^2$.

b) Find the kernel of T, and find its dimension.

Solution We want $p(s) = a + bs + cs^2$ such that $a + \frac{b}{2}t + \frac{c}{3}t^2 = 0$ for all t. Hence $p(s) = 0$ and the kernel is trivial namely its dimension is 0.

c) Find the range (=image) of T, and compute its dimension.

Solution Then image contains elements $a + \frac{b}{2}t + \frac{c}{3}t^2 = a + \hat{b}t + \hat{c}t^2$ for any $a, \hat{b}, \hat{c} \in \mathbb{R}$ hence $\text{im}T = \mathcal{P}_2$, namely the dimension of the image is 3.

d) Verify the dimension of the kernel and the dimension of the image add up to what you would expect.

Solution Indeed $\dim(\text{im}T) + \dim(\text{ker}T) = 3 = \dim \mathcal{P}_2$.

e) Using the standard basis $\{1, t, t^2\}$ for $\mathcal{P}_2$, represent the linear transformation T as a matrix A.

SOLUTION The previous description of T give us that $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$.

f) Using your matrix represention from (e), find $T(p)$ where $p(t) = t - 2$.

SOLUTION $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1/2 \\ 0 \end{pmatrix}$ hence $T(p) = -2 + 1/2t$.

The remaining problems are from the Lecture notes on Vectors

`http://www.math.upenn.edu/~kazdan/312S13/notes/vectors/vectors10.pdf`

9. [p. 8 #5] The origin and the vectors X, Y, and $X+Y$ define a parallelogram whose diagonals have length $X+Y$ and $X-Y$. Prove the *parallelogram law*

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2;$$

This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

SOLUTION: The standard procedure is to express the norm in terms of the inner product and use the usual algebraic rules for the inner product. Thus

$$\|X+Y\|^2 = \langle X+Y,\, X+Y\rangle = \langle X,\, X\rangle + \langle X,\, Y\rangle + \langle Y,\, X\rangle + \langle Y,\, Y\rangle = \langle X,\, X\rangle + 2\langle X,\, Y\rangle + \langle Y,\, Y\rangle,$$

with a similar formula for $\|X-Y\|^2$. After easy algebra, the result is clear.

10. [p. 8 #6] (Math 240 Review)

a) Find the distance from the straight line $\mathcal{L}$: $3x - 4y = 10$ to the origin. [It may help to observe that this line is parallel to the plane $3x - 4y = 0$, whose normal vector is clearly $\vec{N} = (3, -4)$.]

SOLUTION: Note that the equation of the parallel line $\mathcal{L}_0$ through the origin is $3x - 4y = 0$, which we rewrite as $\langle N, X\rangle = 0$, where $N := (3, -4)$ and $X = (x, y)$. Let X_0 be some point on the original line (say $X_0 = (2, -1)$ – although we won't need to be specific), so $\langle N, X_0\rangle = 10$. Then the desired distance D is the same as the distance from X_0 to the line $\mathcal{L}_0$: $\langle N, X\rangle = 0$, through the origin. But the equation for $\mathcal{L}_0$ says the vector N is perpendicular to the line $\mathcal{L}_0$. Thus the distance D is the length of the projection of X_0 in the direction of N, that is,

$$D = \frac{|\langle N, X_0\rangle|}{\|N\|} = \frac{10}{5} = 2.$$

b) Find the distance from the plane $ax + by + cz = d$ to the origin (assume the vector $\vec{N} = (a, b, c) \neq 0$).

SOLUTION: If X_0 is some point on the plane, the equation of this plane is $\langle N, X \rangle = \langle N, X_0 \rangle$. The solution presented in the above special case generalizes immediately to give

$$D = \frac{|\langle N, X_0 \rangle|}{\|N\|} = \frac{|d|}{\|N\|}.$$

11. [p. 8 #8]

a) If X and Y are real vectors, show that

$$\langle X, Y \rangle = \frac{1}{4}\left(\|X+Y\|^2 - \|X-Y\|^2\right). \tag{4}$$

This formula is the simplest way to recover properties of the inner product from the norm.

SOLUTION: The straightforward procedure is the same as in Problem 9: rewrite the norms on the right side of equation (4) in terms of the inner product and expand using algebra.

b) As an application, show that if a square matrix R has the property that it preserves length, so $\|RX\| = \|X\|$ for every vector X, then it preserves the inner product, that is, $\langle RX, RY \rangle = \langle X, Y \rangle$ for all vectors X and Y.

SOLUTION: We know that $\|RZ\| = \|Z\|$ for any vector Z. This implies $\|R(X + Y)\| = \|X+Y\|$ for any vectors X and Y, and, similarly, $\|R(X-Y)\| = \|X-Y\|$ for any vectors X and Y. Consequently, by equation (4) (used twice)

$$\begin{aligned} 4\langle RX, RY \rangle &= \|R(X+Y)\|^2 - \|R(X-Y)\|^2 \\ &= \|X+Y\|^2 - \|X-Y\|^2 \\ &= 4\langle X, Y \rangle \end{aligned}$$

for all vectors X and Y.

12. [p. 9 #10] (Also done in class)

a) If a certain matrix C satisfies $\langle X, CY \rangle = 0$ for *all* vectors X and Y, show that $C = 0$.

SOLUTION: Since X can be *any* vector, let $X = CY$ to show that $\|CY\|^2 = \langle CY, CY \rangle = 0$. Thus $CY = 0$ for all Y so $C = 0$.

b) If the matrices A and B satisfy $\langle X, AY \rangle = \langle X, BY \rangle$ for all vectors X and Y, show that $A = B$.

SOLUTION: We have

$$0 = \langle X, AY \rangle - \langle X, BY \rangle = \langle X, (AY - BY) \rangle = \langle X, (A-B)Y \rangle$$

for all X and Y so by part (a) with $C := A - B$, we conclude that $A = B$.

13. [p. 9 #11] A matrix A is called *anti-symmetric* (or skew-symmetric) if $A^* = -A$.

a) Give an example of a 3×3 anti-symmetric matrix (other than the trivial $A = 0$).

SOLUTION: The most general anti-symmetric 3×3 matrix has the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

b) If A is any anti-symmetric matrix, show that $\langle X, AX \rangle = 0$ for all vectors X.

SOLUTION: The key point is to use the definition of A^* hs having the property $\langle X, AY \rangle = \langle A^*X, Y \rangle$ for all X and Y. This is equivalent to $\langle AX, Y \rangle = \langle X, A^*Y \rangle$ for all X and Y. [Using the fact that for a matrux A^* happens to be the transpose often causes extra confusion.] Thus

$$\langle X, AX \rangle = \langle A^*X, X \rangle = -\langle AX, X \rangle = -\langle X, AX \rangle.$$

so $2\langle X, AX \rangle = 0$ and we are done: $\langle X, AX \rangle = 0$.

[Last revised: February 17, 2014]