Problem Set 5

DUE: In class Thursday, Thurs. Feb. 27. *Late papers will be accepted until 1:00 PM Friday.*

For the coming week, please read Chapter 5, Sections 5., 5.2, 5.3 [*except* for pages 221–223 on the QR Factorization], and Section 5.5.
We will not cover the material on QR factorization. It is an important numerical technique – but our time is short. (We will cover Section 5.4 on the method of Least Squares soon.)

Please reread pages 1–7 in the Lecture notes on Vectors:
`http://www.math.upenn.edu/~kazdan/312S13/notes/vectors/vectors10.pdf`
and read: `http://www.math.upenn.edu//~kazdan/312S14/notes/orthogonal-example.pdf`
on orthogonal projections.

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple exercises. These are *not* to be handed in.

> Sec. 5.1, #28, 29, 31
> Sec. 5.2 #33

1. a) For which values of the constant a and b are the vectors $U = (1+a, -2b, 4)$ and $V = (2, 1, -1)$ perpendicular?

   b) For which values of the constant a, and b is the above vector U, perpendicular to both V and the vector $W = (1, 1, 0)$?

2. [LIKE BRETSCHER, SEC. 5.1 #16] Consider the following orthonormal vectors in $\mathbb{R}^4$

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \qquad \vec{u}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

   a) Let S be the span of $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x} = (1, 2, 3, 4)$. Compute the orthogonal projection, $\text{proj}_S\,\vec{x}$, of $\vec{x}$ into S.

   b) Verify that the vector $\vec{w} := \vec{x} - \text{proj}_S\vec{x}$ is orthogonal to S.

   c) Show that $\|\vec{x}\|^2 = \|\text{proj}_S\vec{x}\|^2 + \|\vec{w}\|^2$.

   d) Compute the distance from $\vec{x}$ to the subspace S.

3. [BRETSCHER, SEC. 5.1 #16] Using the vectors from the previous problem, can you find a vector u_4 in $\mathbb{R}^4$ such that the vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, $\vec{u}_4$ are orthonormal? If so, how many such vectors are there?

4. [Bretscher, Sec. 5.1 #21] Find scalars a, b, c, d, e, f, and g so that the following vectors are orthonormal:

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \quad \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \quad \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}.$$

5. Let V be an inner product space and S a subspace. Then we write $S^{\perp}$ for the set of all vectors in V that are orthogonal to S. It is called the *orthogonal complement* of S, and written $S^{\perp}$. Clearly is also a subspace of V.

   a) In $\mathbb{R}^3$, let S be the points (x_1, x_2, x_3) that satisfy $x_1 - 2x_2 + x_3 = 0$. What is the dimension of $S^{\perp}$? [This should be a simple mental exercise.]

   b) Let $A : \mathbb{R}^3 \to \mathbb{R}^5$. If the dimension of the kernel of A is 2, what is the dimension of image$(A)^{\perp}$?

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6. [Bretscher, Sec. 5.1 #17] In $\mathbb{R}^4$ find a basis for $W^{\perp}$, where

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right\}.$$

7. Here we the linear space $L_2(-1, 1)$ with the usual inner product $\langle f,\, g \rangle := \int_{-1}^{1} f(x)g(x)\, dx$ (assuming $f(x)$ and $g(x)$ are integrable). A function $f(x)$ is called an *even function* if $f(-x) = f(x)$. An example is $f(x) := 2 - 7x^6$. Similarly, $f(x)$ is *odd* if $f(-x) = -f(x)$. An example is $f(x) = 2x - \sin 3x$. The function $f(x) = 1 - 2x$ is neither even nor odd.

   a) If $h(x)$ is any odd (integrable) function show that $\int_{-1}^{1} h(x)\, dx = 0$.

   b) Show that any even function $f(x)$ and any odd function $g(x)$ are orthogonal.

   c) In this inner product, show that $\cos 3x$ and $\sin 8x$ are orthogonal.

   d) Given any function $f(x)$ show there is a unique even function $f_{\text{even}}(x)$ and an odd function $f_{\text{odd}}(x)$ so that

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x).$$

   Find this decomposition for $f(x) = e^x$.

   e) Continuing from the previous part, show that

$$\int_{-1}^{1} f(x)^2\, dx = \int_{-1}^{1} f_{\text{even}}(x)^2\, dx + \int_{-1}^{1} f_{\text{odd}}(x)^2\, dx,$$

that is,

$$\|f\|^2 = \|f_{\text{even}}\|^2 + \|f_{\text{odd}}\|^2$$

f) Compute $\displaystyle\int_{-1}^{1} \left[3 + 5x^7 + 2x\cos x - \frac{3x}{1+x^4} + xe^{\cos 2x}\right] dx\,.$

8. [Bretscher, Sec. 5.5 #24]. Using the inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)\,dx$, for certain polynomials $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ say we are given the following table of inner products:

$\langle\ ,\ \rangle$	$\mathbf{f}$	$\mathbf{g}$	$\mathbf{h}$
$\mathbf{f}$	4	0	8
$\mathbf{g}$	0	1	3
$\mathbf{h}$	8	3	50

For example, $\langle \mathbf{g}, \mathbf{h}\rangle = \langle \mathbf{h}, \mathbf{g}\rangle = 3$. Let E be the span of $\mathbf{f}$ and $\mathbf{g}$.

a) Compute $\langle \mathbf{f}, \mathbf{g}+\mathbf{h}\rangle$.

b) Compute $\|\mathbf{g}+\mathbf{h}\|$.

c) Find $\text{proj}_E\mathbf{h}$. [Express your solution as linear combinations of $\mathbf{f}$ and $\mathbf{g}$.]

d) Find an orthonormal basis of the span of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ [Express your results as linear combinations of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$.]

9. Let V be the linear space of 4×4 matrices with real entries. Define a linear transformation $L : V \to V$ by the rule $L(A) = \frac{1}{2}(A + A^T)$. [Here A^T is the matrix transpose of A.]

a) Verify that L is linear.

b) Describe the image of L and find it's dimension. [Try the case of 2×2 matrices first.]

c) Verify that the rank and nullity add up to what you would expect. [Note: This map L is called the *symmetrization operator*.]

d) Given any 4×4 matrix A, find a symmetric matrix A_s and an anti-symmetric A_a so that $A = A_s + A_a$. [You should find simple formulas for A_s and A_a in terms of A and $A*$.]

10. a) For $\vec{x}\in\mathbb{R}^2$, let $Q(\vec{x}) = 3x_1^2 + 2x_1x_2 - 5x_2^2$. Find a *symmetric* matrix A so that $Q(\vec{x}) = \langle \vec{x}, A\vec{x}\rangle$. Can you find some different symmetric matrix A? Why or why not?

b) For $\vec{x}\in\mathbb{R}^3$, let $Q(\vec{x}) = 3x_1^2 + 2x_1x_2 - 5x_x^2 - 4x_1x_3 + 2x_3^2$. Find a *symmetric* 3×3 matrix A so that $Q(\vec{x}) = \langle \vec{x}, A\vec{x}\rangle$.

Could you have found some different symmetric matrix A?

11. Let $A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$. If $\langle \vec{x},\, A\vec{x} \rangle > 0$ for all $\vec{x} = (x_1,\, x_2,\, x_3,\, x_4) \in \mathbb{R}^4,\ \vec{x} \neq 0$, show that $a,\, b,\, c,\, d$ must all be positive.

12. The following problem concerns the *correlation coefficient* (p. 213 in Bretscher).

a) Say you have a table of data. The first column, the vector $V = (v_1,\, \ldots, v_n)$, is the number of hours each student studied for an exam, the second column, $W = (w_1,\, \ldots, w_n)$, is the list of corresponding grades on the exam ($A = 4,0,\ B = 3.0$, etc.). To compute with data effectively, we should normalize by subtracting the averages (mean) $\bar{v} = (v_1 + \cdots + v_n)/n$ and $\bar{w} = (w_1 + \cdots + w_n)/n)$ to get the normalized data vectors

$$V_{\text{norm}} := (v_1 - \bar{v},\, \ldots, v_n - \bar{v}), \qquad W_{\text{norm}} := (w_1 - \bar{w},\, \ldots, w_n - \bar{w})$$

(we could further normalize to make both of these to be unit vectors, but the definition of the recurssion coefficient does this for us).

What would you roughly anticipate the correlation coefficient of the normalized data will tell us? Why?

b) This time there is a trial of the effectiveness of a new medication. There are n people, all of whom have a certain disease. Some are given the new drug, some a placebo. The corresponding data vector $V = (v_1, \ldots, v_n)$ with a component being either 1 (patient given the test drug), or 0 (given a placebo).

After several months the medication is evaluated resulting in a data vector $W = (w_1, \ldots, w_n)$ where $-1 \leq w_j \leq 1$ is determined using the following guidelines

$$w_j = \begin{cases} +1 & \text{if the } j^{th} \text{ patient has been cured,} \\ 0 & \text{if the } j^{th} \text{ patient is essentially unchanged, .} \\ -1 & \text{if the } j^{th} \text{ patient has died.} \end{cases}$$

After normalizing the data vectors, you compute the correlation coefficient r.

If $r = +0.8$, what would you conclude?
If $r = -0.2$, what would you conclude?
If $r = -0.7$, what would you conclude?

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let P_1, P_2, $\ldots$, P_k be points in $\mathbb{R}^n$. For $X \in \mathbb{R}^n$ let

$$Q(X) := \|X - P_1\|^2 + \|X - P_2\|^2 + \cdots \|X - P_k\|^2.$$

Determine the point X that minimizes $Q(X)$.

B-2 Consider the space $C_0^2[0,1]$ of twice continuously differentiable functions $u(x)$ with $u(0) = 0$ and $u(1) = 0$. Define the differential operator Mu by the formula $M : u = ((1 + x^2)u')'$. Find the adjoint M^* (you should find that M is self-adjoint).

[See `http://www.math.upenn.edu/~kazdan/312S13/notes/Lu=-DDu.pdf`]

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