Math 312, Spring 2014 Jerry L. Kazdan

# Problem Set 5 Solutions

DUE: In class Thursday, Thurs. Feb. 27. *Late papers will be accepted until 1:00 PM Friday.*

For the coming week, please read Chapter 5, Sections 5., 5.2, 5.3 [*except* for pages 221–223 on the QR Factorization], and Section 5.5.
We will not cover the material on QR factorization. It is an important numerical technique – but our time is short. (We will cover Section 5.4 on the method of Least Squares soon.)

Please reread pages 1–7 in the Lecture notes on Vectors:
`http://www.math.upenn.edu/~kazdan/312S13/notes/vectors/vectors10.pdf`
and read:
`http://www.math.upenn.edu//~kazdan/312S14/notes/orthogonal-example.pdf`
on orthogonal projections.

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple exercises. These are *not* to be handed in.

Sec. 5.1, #28, 29, 31
Sec. 5.2 #33

1. a) For which values of the constant $a$ and $b$ are the vectors $U = (1 + a, -2b, 4)$ and $V = (2, 1, -1)$ perpendicular?

   b) For which values of the constant $a$, and $b$ is the above vector $U$, perpendicular to both $V$ and the vector $W = (1, 1, 0)$?

   SOLUTION

   a) We want $< U, V >= 0$, i.e. $2+2a-2b-4 = 0$, so for any $a, b$ such that $a-b = 1$, $U, V$ are perpendicular.

   b) We also need $0 =< U, W >= 1+a-2b$. So we solve the system of the two equations and obtain $a = 3,\ b = 2$.

2. [LIKE BRETSCHER, SEC. 5.1 #16] Consider the following orthonormal vectors in $\mathbb{R}^4$

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \qquad \vec{u}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

   a) Let $S$ be the span of $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x} = (1, 2, 3, 4)$. Compute the orthogonal projection, $\text{proj}_S\,\vec{x}$, of $\vec{x}$ into S.

   b) Verify that the vector $\vec{w} := \vec{x} - \text{proj}_S\vec{x}$ is orthogonal to $S$.

   c) Show that $\|\vec{x}\|^2 = \|\text{proj}_S\vec{x}\|^2 + \|\vec{w}\|^2$.

d) Compute the distance from $\vec{x}$ to the subspace $S$.

SOLUTION

a) Since $u_1, u_2$ unit vectors we have that $\text{proj}_S \vec{x} = < x, \vec{u}_1 > \vec{u}_1 + < x, \vec{u}_2 > \vec{u}_2 = 5\vec{u}_1 - 2\vec{u}_2 = (3/2, 3/2, 7/2, 7/2)$.

b) $\vec{w} = (-1/2, 1/2, -1/2, 1/2)$ so $< \vec{w}, \vec{u}_1 >= 0$ and $< \vec{w}, \vec{u}_2 >= 0$ which proves what we want.

c) $\|\text{proj}_S \vec{x}\|^2 + \|\vec{w}\|^2 = 29 + 1 = 30 = \|\vec{x}\|^2$.

d) The distance from $\vec{x}$ to $S$ is the norm of $\vec{w}$ which is equal to $1$.

3. [BRETSCHER, SEC. 5.1 #16] Using the vectors from the previous problem, can you find a vector $u_4$ in $\mathbb{R}^4$ such that the vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, $\vec{u}_4$ are orthonormal? If so, how many such vectors are there?

SOLUTION Since $\mathbb{R}^4$ is four dimensional, you can extend these three orthonormal vectors to an orthonormal basis. First find a vector orthogonal to these three; then normalize it to be a unit vector. In this case,

SIMPLEST: In the previous problem you already found a vector $\vec{w}$, orthogonal to $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$. Use it. Moreover, it already happens to be a unit vector so let $\vec{u}_4 = \vec{w}$ to obtain the desired orthonormal basis.

ALTERNATE 1: In this particular example you might immediately guess the fourth basis vector:

$$\vec{u}_4 = \pm \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \tag{1}$$

These two (note the $\pm$) are the only possibility since the orthogonal complement of the span of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ is one dimensional so a basis will have only one vector. After we have found one, which we call $\vec{u}_4$, any other, say $\hat{\vec{w}}$ must have the form $\vec{w} = c\vec{u}_4$ for some constant $c$. Because we want a unit vector,

$$1 = \|\vec{w}\| = c^2\|\vec{u}_4\| = c^2,$$

so $c = \pm 1$.

ALTERNATE 2: But what if this $\vec{u}_4$ didn't immediately come to mind? Use the Gram-Schmidt process. Pick any vector *not* in the span of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$. Almost any vector in $\mathbb{R}^4$ will do. I will try the simple $\vec{w} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. We want to write $\vec{w}$ in the form

$$\vec{w} = a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + \vec{z},$$

where $\vec{z}$ is orthogonal to $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$. Taking the inner product of both sides of this successively with $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ (which are unit vectors), we find that

$$a = \langle \vec{w}, \vec{u}_1 \rangle = 1/2, \qquad b = \langle \vec{w}, \vec{u}_2 \rangle = 1/2, \qquad c = \langle \vec{w}, \vec{u}_3 \rangle = 1/2.$$

Then

$$\vec{z} = \vec{w} - [(1/2)\vec{u}_1 + (1/2)\vec{u}_2 + (1/2)\vec{u}_3] = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{pmatrix}$$

To get the desired unit vector we let $\vec{u}_4 = \vec{z}/\|\vec{z}\|$ which agrees with (1)

4. [Bretscher, Sec. 5.1 #21] Find scalars $a$, $b$, $c$, $d$, $e$, $f$, and $g$ so that the following vectors are orthonormal:

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \qquad \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \qquad \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}.$$

Solution The orthogonality gives

$$ab + d + fg = 0, \qquad ac + ed + f/2 = 0, \qquad bc + e + g/2 = 0.$$

Because we want unit vectors, so we can't scale the second or third vectors, we need $b = g = 0$ and we can't simply let $c = e = 0$ (it took me a few minutes to grasp this). The orthogonality conditions are then

$$d = 0, \qquad ac + f/2 = 0, \qquad e = 0.$$

That these are unit vectors gives $a^2 + f^2 = 1$ and $c^2 + 1/4 = 1$. Therefore $c = \pm\sqrt{3}/2$, so $f = \mp(\sqrt{3})a$, which in turn implies $a = \pm 1/2$.

5. Let $V$ be an inner product space and $S$ a subspace. Then we write $S^\perp$ for the set of all vectors in $V$ that are orthogonal to $S$. It is called the *orthogonal complement* of $S$, and written $S^\perp$. Clearly is also a subspace of $V$.

   a) In $\mathbb{R}^3$, let $S$ be the points $(x_1, x_2, x_3)$ that satisfy $x_1 - 2x_2 + x_3 = 0$. What is the dimension of $S^\perp$? [This should be a simple mental exercise.]

   b) Let $A : \mathbb{R}^3 \to \mathbb{R}^5$. If the dimension of the kernel of $A$ is 2, what is the dimension of $\text{image}(A)^\perp$?

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Solution

a) The plane $S$ is a 2-dimensional linear subspace of $\mathbb{R}^3$, hence $S^\perp$ has dimension 1.

b) $\dim(\text{image}(A)^\perp) = \dim \mathbb{R}^5 - \dim(\text{image}(A)) = \dim \mathbb{R}^5 - (\dim \mathbb{R}^3 - \dim(\ker(A)) = 5 - 1 = 4$.

6. [Bretscher, Sec. 5.1 #17] In $\mathbb{R}^4$ find a basis for $W^\perp$, where

$$W = \text{span}\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}5\\6\\7\\8\end{pmatrix}\right\}.$$

Solution The vectors $\vec{x} = (x_1,\, x_2,\, x_3,\, x_4) \in W^\perp$ must satisfy

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= 0 \end{aligned}$$

Solving these equations for $x_1$ and $x_2$ in terms of $x_3$ and $x_4$ we find

$$x_1 = x_3 + 2x_4 \qquad x_2 = -2x_3 - 3x_4.$$

Thus

$$\vec{x} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix}1\\-2\\1\\0\end{pmatrix} x_3 + \begin{pmatrix}2\\-3\\0\\1\end{pmatrix} x_4$$

The two vectors at the end of the previous line are a basis for $W^\perp$. (They are not an orthogonal basis.)

7. Here we the linear space $L_2(-1,1)$ with the usual inner product $\langle f,\, g\rangle := \int_{-1}^{1} f(x)g(x)\,dx$ (assuming $f(x)$ and $g(x)$ are integrable). A function $f(x)$ is called an *even function* if $f(-x) = f(x)$. An example is $f(x) := 2 - 7x^6$. Similarly, $f(x)$ is *odd* if $f(-x) = -f(x)$. An example is $f(x) = 2x - \sin 3x$. The function $f(x) = 1 - 2x$ is neither even nor odd.

a) If $h(x)$ is any odd (integrable) function show that $\int_{-1}^{1} h(x)\,dx = 0$.

b) Show that any even function $f(x)$ and any odd function $g(x)$ are orthogonal.

c) In this inner product, show that $\cos 3x$ and $\sin 8x$ are orthogonal.

d) Given any function $f(x)$ show there is a unique even function $f_{\text{even}}(x)$ and an odd function $f_{\text{odd}}(x)$ so that

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x). \tag{2}$$

Find this decomposition for $f(x) = e^x$.

e) Continuing from the previous part, show that

$$\int_{-1}^{1} f(x)^2\,dx = \int_{-1}^{1} f_{\text{even}}(x)^2\,dx + \int_{-1}^{1} f_{\text{odd}}(x)^2\,dx,$$

that is,

$$\|f\|^2 = \|f_{\text{even}}\|^2 + \|f_{\text{odd}}\|^2$$

f) Compute $\displaystyle\int_{-1}^{1}\left[3 + 5x^7 + 2x\cos x - \frac{3x}{1+x^4} + xe^{\cos 2x}\right] dx\,.$

SOLUTION

a) Note $\int_{-1}^{0} h(x)\,dx = -\int_{1}^{0} h(-t)\,dt = -\int_{0}^{1} h(x)\,dx = 0$. The assertion is now clear.

b) Since $f$ even, $g$ odd we have $fg$ odd hence $< f, g >= 0$ from part (a).

c) Since cos is an even function and sin is an odd function we obtain from part (b) that they are orthogonal.

d) Assume there are such functions $f_{\text{even}}(x)$ and $f_{\text{odd}}(x)$. Then from equation (2),

$$f(-x) = f_{\text{even}}(-x) + f_{\text{odd}}(-x) = f_{\text{even}}(x) - f_{\text{odd}}(x).$$

Adding this equation to (2) gives $f_{even} = \frac{1}{2}(f(x) + f(-x))$, while subtracting them gives $f_{odd} = \frac{1}{2}(f(x) - f(-x))$. These are the unique solutions of equation (2).

For $e^x$, $(e^x)_{even} = \frac{e^x + e^{-x}}{2} = \cosh x$, $(e^x)_{odd} = \frac{e^x - e^{-x}}{2} = \sinh x$.

e) This is immediate from part (d) since $< f_{even}, f_{odd} >= 0$.

f) Notice that the $5x^7 + 2x\cos x - \frac{3x}{1+x^4} + xe^{\cos 2x}$ is an odd function hence the integral is equal to $\int_{-1}^{1} 3\,dx = 6$.

8. [BRETSCHER, SEC. 5.5 #24]. Using the inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)\,dx$, for certain polynomials $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ say we are given the following table of inner products:

| $\langle\ ,\ \rangle$ | $\mathbf{f}$ | $\mathbf{g}$ | $\mathbf{h}$ |
|---|---|---|---|
| $\mathbf{f}$ | 4 | 0 | 8 |
| $\mathbf{g}$ | 0 | 1 | 3 |
| $\mathbf{h}$ | 8 | 3 | 50 |

For example, $\langle \mathbf{g}, \mathbf{h}\rangle = \langle \mathbf{h}, \mathbf{g}\rangle = 3$. Let $E$ be the span of $\mathbf{f}$ and $\mathbf{g}$.

a) Compute $\langle \mathbf{f}, \mathbf{g} + \mathbf{h}\rangle$.

SOLUTION: $\langle \mathbf{f}, \mathbf{g} + \mathbf{h}\rangle = 0 + 8 = 8$.

b) Compute $\|\mathbf{g} + \mathbf{h}\|$.

SOLUTION: $\|\mathbf{g} + \mathbf{h}\|^2 = 1 + 2\cdot 3 + 50 = 57$ so $\|\mathbf{g} + \mathbf{h}\| = \sqrt{57}$

c) Find $\text{proj}_E\mathbf{h}$. [Express your solution as linear combinations of $\mathbf{f}$ and $\mathbf{g}$.]

SOLUTION: Since $\mathbf{f}$ and $\mathbf{g}$ are orthogonal, they are an orthogonal basis for $E$. Thus $\text{proj}_E\mathbf{h} = a\mathbf{f} + b\mathbf{g}$ for some constants $a$ and $b$, that is,

$$\mathbf{h} = a\mathbf{f} + b\mathbf{g} + \mathbf{w}, \tag{3}$$

for some $\mathbf{w} \perp E$. To find $a$ and $b$, as usual we take the inner product of both sides with $\mathbf{f}$ and $\mathbf{g}$ and get

$$a = \frac{\langle \mathbf{h}, \mathbf{f}\rangle}{\|\mathbf{f}\|^2} = \frac{8}{4} = 2, \qquad b = \frac{\langle \mathbf{h}, \mathbf{g}\rangle}{\|\mathbf{g}\|^2} = \frac{3}{1} = 3.$$

Therefore,

$$\text{proj}_E\mathbf{h} = 2\mathbf{f} + 3\mathbf{g}$$

d) Find an orthonormal basis of the span of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ [Express your results as linear combinations of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$.]

SOLUTION: Since $\mathbf{f}$ and $\mathbf{g}$ are orthogonal and, from equation (3), $\mathbf{w}$ is orthogonal to both $\mathbf{f}$ and $\mathbf{g}$, we find that $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{w}$ are an orthogonal bases. To get an orthonormal basis we need only normalize these. From (3),

$$\|\mathbf{h}\|^2 = \|2\mathbf{f}\|^2 + \|3\mathbf{g}\|^2 + \|\mathbf{w}\|^2$$

so $\|\mathbf{w}\|^2 = 50 - 4\cdot 4 - 9\cdot 1 = 25$. Therefore an orthonormal basis is

$$e_1 := \tfrac{1}{2}\mathbf{f}, \qquad e_2 := \mathbf{g}, \qquad e_3 := \tfrac{1}{5}\mathbf{w} = \tfrac{1}{5}(\mathbf{h} - 2\mathbf{f} - 3\mathbf{g}).$$

9. Let $V$ be the linear space of $4\times 4$ matrices with real entries. Define a linear transformation $L : V \to V$ by the rule $L(A) = \frac{1}{2}(A + A^T)$. [Here $A^T$ is the matrix transpose of $A$.]

a) Verify that $L$ is linear.

SOLUTION: Linearity is a consequence of $(A+B)^T = A^T + B^T$ and $(cA)^T = c(A^T)$.

b) Describe the image of $L$ and find it's dimension. [Try the case of $2\times 2$ matrices first.]

SOLUTION: In the $2\times 2$ case, say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,then $\frac{1}{2}(A + A^T) = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix}$.
So $\operatorname{im} L = \left\{ \begin{pmatrix} a & \beta \\ \beta & d \end{pmatrix} \mid a, \beta, d \in \mathbb{R} \right\}$ and it's 3 dimensional. The key observation is that $A + A^T$ is a symmetric matrix.
In the $4\times 4$ case $\frac{1}{2}(A + A^T)$is the most general $4\times 4$ symmetric matrix:

$$\tfrac{1}{2}(A + A^T) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}.$$

The dimension of this space is $4 + 3 + 2 + 1 = 10$.

c) Verify that the rank and nullity add up to what you would expect. [NOTE: This map $L$ is called the *symmetrization operator*.]

SOLUTION: For 2×2 matrices, $\ker L = \{\begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} = \{\begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} \mid \gamma \in \mathbb{R}\}$ which is 1 dimensional.

For $4 \times 4$ matrices, the kernel is any $4 \times 4$ anti-symmetric matrix:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

The dimension of these is $3+2+1=6$. Note $10+6=16$, which agrees with the dimension of all $4 \times 4$ matrices.

d) Given any $4 \times 4$ matrix $A$, find a symmetric matrix $A_s$ and an anti-symmetric $A_a$ so that $A = A_s + A_a$. [You should find simple formulas for $A_s$ and $A_a$ in terms of $A$ and $A*$.]

SOLUTION: [This is almost identical to Problem 7d) above.] The sought formula

$$A = A_s + A_a \qquad \text{implies} \qquad A^* = A_s - A_a.$$

Adding and subtracting these gives

$$A_s = \tfrac{1}{2}(A + A^T) \qquad \text{and} \qquad A_a = \tfrac{1}{2}(A - A^T).$$

This works for any $n \times n$ matrix.

10. a) For $\vec{x} \in \mathbb{R}^2$, let $Q(\vec{x}) = 3x_1^2 + 2x_1x_2 - 5x_2^2$. Find a *symmetric* matrix $A$ so that $Q(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle$. Can you find some different symmetric matrix $A$? Why or why not?

SOLUTION: $A = \begin{pmatrix} 3 & 1 \\ 1 & -5 \end{pmatrix}$. This is the only symmetric matrix. To see this, use Problem 9d) above to decompose any matrix $A$ as $A = A_s + A_a$ and note (Homework Set 4 # 13) that for any anti-symmetric $A_a$ we have $\langle \vec{x}, A_a\vec{x} \rangle = 0$. Thus for any matrix $A$ we have $\langle \vec{x}, A\vec{x} \rangle = \langle \vec{x}, A_s\vec{x} \rangle$.

b) For $\vec{x} \in \mathbb{R}^3$, let $Q(\vec{x}) = 3x_1^2 + 2x_1x_2 - 5x_2^2 - 4x_1x_3 + 2x_3^2$. Find a *symmetric* $3 \times 3$ matrix $A$ so that $Q(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle$.

Could you have found some different symmetric matrix $A$?

SOLUTION: $A = \begin{pmatrix} 3 & 1 & -2 \\ 1 & -5 & 0 \\ -2 & 0 & 2 \end{pmatrix}$. This symmetric $A$ is unique, just as in part a).

11. Let $A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$. If $\langle \vec{x}, A\vec{x} \rangle > 0$ for all $\vec{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $\vec{x} \neq 0$, show that $a$, $b$, $c$, $d$ must all be positive.

SOLUTION: Making the calculations we obtain $ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 > 0$ for $\vec{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Letting $\vec{x} = \vec{e}_i$, $i = 1, 2, 3, 4$ where $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$ we obtain what we want.

12. The following problem concerns the *correlation coefficient* (p. 213 in Bretscher).

a) Say you have a table of data. The first column, the vector $V = (v_1, \ldots, v_n)$, is the number of hours each student studied for an exam, the second column, $W = (w_1, \ldots, w_n)$, is the list of corresponding grades on the exam ($A = 4,0$, $B = 3.0$, etc.). To compute with data effectively, we should normalize by subtracting the averages (mean) $\bar{v} = (v_1 + \cdots + v_n)/n$ and $\bar{w} = (w_1 + \cdots + w_n)/n$) to get the normalized data vectors

$$V_{\text{norm}} := (v_1 - \bar{v}, \ldots, v_n - \bar{v}), \qquad W_{\text{norm}} := (w_1 - \bar{w}, \ldots, w_n - \bar{w})$$

(we could further normalize to make both of these to be unit vectors, but the definition of the recurssion coefficient does this for us).

What would you roughly anticipate the correlation coefficient of the normalized data will tell us? Why?

SOLUTION: It will tell us what is the relationship between how many hours a student studies and what grade he achieved. If $r$ is positive this means that studying many hours has a positive impact on the grade and as near $r$ is to the value 1, this means greater impact. If $r$ is negatice this means that studying many hours has a negative impact on the grade and as near $r$ is to the value $-1$, this means greater (negative) impact.

The value of $r$ indicate how many of the corresponding variables of the normalized data have the same or different sign and from this we can obtain conclusions regarding the behavior of the one data vector in comparison with the other data vector.

b) This time there is a trial of the effectiveness of a new medication. There are $n$ people, all of whom have a certain disease. Some are given the new drug, some a placebo. The corresponding data vector $V = (v_1, \ldots, v_n)$ with a component being either 1 (patient given the test drug), or 0 (given a placebo).

After several months the medication is evaluated resulting in a data vector $W = (w_1, \ldots, w_n)$ where $-1 \leq w_j \leq 1$ is determined using the following guidelines

$$w_j = \begin{cases} +1 & \text{if the } j^{th} \text{ patient has been cured,} \\ 0 & \text{if the } j^{th} \text{ patient is essentially unchanged,} \\ -1 & \text{if the } j^{th} \text{ patient has died.} \end{cases}.$$

After normalizing the data vectors, you compute the correlation coefficient $r$.

If $r = +0.8$, what would you conclude?

If $r = -0.2$, what would you conclude?

If $r = -0.7$, what would you conclude?

SOLUTION: If $r = 0.8$ this means that it is near the value 1 which means that out of those that took the test drug most of them were cured. If $r = -0.2$ this means that out of those that took the test drug most of them were unchanged or died. If $r = -0.7$ then this means that out of those that took the test drug most of them died.

## Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let $P_1$, $P_2$, ..., $P_k$ be points in $\mathbb{R}^n$. For $X \in \mathbb{R}^n$ let

$$Q(X) := \|X - P_1\|^2 + \|X - P_2\|^2 + \cdots \|X - P_k\|^2.$$

Determine the point $X$ that minimizes $Q(X)$.

B-2 Consider the space $C_0^2[0,1]$ of twice continuously differentiable functions $u(x)$ with $u(0) = 0$ and $u(1) = 0$. Define the differential operator $Mu$ by the formula $M : u = ((1 + x^2)u')'$. Find the adjoint $M^*$ (you should find that $M$ is self-adjoint).

[See `http://www.math.upenn.edu/~kazdan/312S13/notes/Lu=-DDu.pdf`]

[Last revised: March 9, 2014]