Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 6

DUE: In class Thurs, March 6. *Late papers will be accepted until 1:00 PM Friday.*

For the coming week, please read Sections 5.3-5.4 and the notes
`http://www.math.upenn.edu/~kazdan/312S13/notes/vectors/vectors10.pdf`
on Vectors and Least Squares.

1. [BRETSCHER, SEC. 5.1 #26] Find the orthogonal projection P_S of $\vec{x} := \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix}$ into the subspace S of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

2. [BRETSCHER, SEC. 5.4 #2] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$. Find a basis for $\ker A^*$.

   Draw a sketch illustrating the formula $(\operatorname{im} A)^\perp = \ker A^*$ in this case.

3. [BRETSCHER, SEC. 5.4 #16] Let A be an $n \times k$ matrix. Show that

$$\operatorname{rank} A = \operatorname{rank} A^*.$$

4. [BRETSCHER, SEC. 5.2 #32] Find an orthonormal basis for the plane $x_1 + x_2 + x_3 = 0$.

5. [BRETSCHER, SEC. 5.3 #10] Consider the space $\mathcal{P}_\in$ of real polynomials of degree at most 2 with the inner product

$$\langle f, g \rangle = \tfrac{1}{2} \int_{-1}^{1} f(t)g(t)\, dt.$$

   Find an orthonormal basis for all the functions in $\mathcal{P}_2$ that are orthogonal to $f(t) = t$.

6. [BRETSCHER, SEC. 5.3 #16] Consider the space $\mathcal{P}_1$ with the inner product

$$\langle f, g \rangle = \int_{0}^{1} f(t)g(t)\, dt.$$

   a) Find an orthonormal basis for this space. [Suggestion: Let $e_1(t = 1$ and pick $e_2(t) = a + bt$ to be orthogonal to e_1.]

b) Find the linear polynomial $g(t) = a + bt$ that best approximates the polynomial $f(t) = t^2$. Thus, one wants to pick $g(t)$ so that $\|f - g\|$ is as small as possible. [Question: In an inner product space V, if you have a subspace $S \subset V$ and a vector $\vec{y} \in V$, how can you find the vector in S that is closest to $\vec{y}$?]

7. Let $f(x) := \begin{cases} 0 & \text{if } -\pi \le x \le -\pi/2 \\ 1 & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 \le x \le \pi \end{cases}$ and define $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Find the Fourier Series of $f(x)$.

8. [Bretscher, Sec. 5.1 #37] Consider a plane V in $\mathbb{R}^3$ with orthonormal basis $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x}$ be a vector in $\mathbb{R}^3$. Find a formula for the reflection $R\vec{x}$ of $\vec{x}$ across the plane V. Your answer will involve $P_V\vec{x}$, the orthogonal projection of $\vec{x}$ into the plane V. [Suggestion: Use that $(I - P_V)\vec{x}$ is the component of $\vec{x}$ that is orthogonal to V. In a reflection, this is the part of $\vec{x}$ that is flipped.]

9. Let V be a linear space with an inner product and $P : V \to V$ a linear map. P is called a *projection* if $P^2 = P$. Let $Q := I - P$.
   a) Show that $Q^2 = Q$, so Q is also a projection. Show that the image of P is the kernel of Q.
   b) A projection P is called an *orthogonal projection* if the image of P is orthogonal to the kernel of P. If $P = P^*$, show that P is an orthogonal projection.
   c) Conversely, if P is an orthogonal projection, show that $P = P^*$.

10. Let A be a real matrix, not necessarily square.
   a) If A is onto, show that A^* is one-to-one.
   b) If A is one-to-one, show that A^* is onto.

11. Let A be a real matrix, not necessarily square.
   a) Show that both A^*A and AA^* are self-adjoint.
   b) Show that $\ker A = \ker A^*A$. [Hint: Show separately that $\ker A \subset \ker A^*A$ and $\ker A \supset \ker A^*A$. The identity $\langle \vec{x}, A^*A\vec{x}\rangle = \langle A\vec{x}, A\vec{x}\rangle$ is useful.]
   c) If A is one-to-one, show that A^*A is invertible
   d) If A is onto, show that AA^* is invertible.

12. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map that is onto but not one-to-one. Say X_1 is a solution of $AX = Y$. Is there a "best" possible solution? What can one say? Think about this before reading the next paragraph.

Show that there is exactly one solution X_2 of the form $X_2 = A^*V$ for some V, so $AA^*V = Y$. Moreover of all the solutions X of $AX = Y$, show that X_2 is closest to the origin.

Quadratic Polynomials Using Inner Products

If A is a real symmetric matrix (so it is self-adjoint), then $Q(\vec{x}) := \langle \vec{x},\, A\vec{x}\rangle$ is a quadratic polynomial. Given a quadratic polynomial, it is easy to find the (unique) symmetric symmentic matrix A. Here is an example. Say $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2$ To find A, note that $-8x_1x_2 = -4x_1x_2 - 4x_2x_2$ so we can rewite Q as

$$Q(\vec{x}) := 3x_1^2 - 4x_1x_2 - 4x_2x_1 - 5x_2^2.$$

If we let

$$A := \begin{pmatrix} 3 & -4 \\ -4 & -5 \end{pmatrix} \qquad \text{[Note } A \text{ is a symmetric matrix]},$$

then it is easy to verify that $Q(\vec{x}) = \langle \vec{x},\, A\vec{x}\rangle$. In the remaining problems we will use this to help work with quadratic polynomials.

13. In each of these find a 3×3 *symmetric* matrix A so that $Q(\vec{x}) = \langle \vec{x},\, A\vec{x}\rangle$.
    a) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 + x_3^2$.
    b) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3 + x_3^2$.
    c) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3$.

14. [Lower order terms and Completing the Square] Which is simpler:

$$z = x_1^2 + 4x_2^2 - 2x_1 + 4x_2 + 2 \qquad \text{or} \qquad z = y_1^2 + 4y_2^2 \text{ ?}$$

If we let $y_1 = x_1 - 1$ and $y_2 = x_2 + 1/2$, they are essentially the same. All we did was translate the origin to $(1, -1/2)$.

The point of this problem is to generalize this to quadratic polynomials in several variables. Let

$$\begin{aligned} Q(\vec{x}) &= \sum a_{ij}x_ix_j + 2\sum b_ix_i + c \\ &= \langle \vec{x},\, A\vec{x}\rangle + 2\langle b,\, \vec{x}\rangle + c \end{aligned}$$

be a real quadratic polynomial so $\vec{x} = (x_1, \dots, x_n)$, $\vec{b} = (b_1, \dots, b_n)$ are real vectors and $A = (a_{ij})$ is a real symmetric $n \times n$ matrix.

In the case $n = 1$, $Q(x) = ax^2 + 2bx + c$ which is clearly simpler in the special case $b = 0$. In this case, if $a \neq 0$, by completing the square we find

$$Q(x) = a\,(x + b/a)^2 + c - 2b^2/a = ay^2 + \gamma,$$

where we let $y = x - b/a$ and $\gamma = c - b^2/a$. Thus, by translating the origin: $x = y + b/a$ we can eliminate the linear term in the quadatratic polynomial – so it becomes simpler.

a) Similarly, for any dimension n, if A is invertible, using the above as a model, show there is a change of variables $\vec{y} == \vec{x} - \vec{v}$ (this is a translation by the vector $\vec{v}$) so that in the new $\vec{y}$ variables Q hasthe form

$$\hat{Q}(\vec{y}) := Q(\vec{y} + \vec{v}) = \langle \vec{y}, A\vec{y} \rangle + \gamma \quad \text{that is,} \quad \hat{Q}(\vec{y}) = \sum a_{ij} y_i y_j + \gamma,$$

where γ involves A, b, and c – but no terms that are linear in $\vec{y}$. [In the case $n = 1$, which you should try *first*, this means using a change of variables $y = x - v$ to change the polynomial $ax^2 + 2bx + c$ to the simpler $ay^2 + \gamma$.]

b) As an example, apply this to $Q(\vec{x}) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4$.

15. For $\vec{x} \in \mathbb{R}^n$ let $Q(\vec{x}) := \langle \vec{x}, A\vec{x} \rangle$, where A is a real symmetric matrix. We say that A is *positive definite* if $Q(\vec{x}) > 0$ for all $\vec{x} \neq 0$, *negative definite* if $Q(\vec{x}) < 0$ for all $\vec{x} \neq 0$, and *indefinite* if $Q(\vec{x}) > 0$ for some $\vec{x}$ but $Q(\vec{x}) < 0$ for some other $\vec{x}$.

a) In the special case $n = 2$ give (simple!) examples of matrices A that are positive definite, negative definite, and indefinite.

b) In the special case where A is an invertible *diagonal* matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

under what conditions is $Q(\vec{x})$ positive definite, negative definite, and indefinite? [Remark: We will see that the general case can *always* be reduced to this special case where A is diagonal.]

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let $\mathcal{S} := \{u(x) \in C^2[0, \pi]$ with $u(0) = u(\pi) = 0\}$ and let $Lu := -u''(x)$. Use the inner product $\langle u, v \rangle = \int_0^\pi u(x)v(x)\, dx$.

a) If u and v are in $\mathcal{S}$, show that $\langle Lu, v \rangle = \langle u, Lv \rangle$. This shows that L is self-adjoint on this space of functions. [Hint: Integrate by parts.]

b) If $u(x) \in \mathcal{S}$, $u \not\equiv 0$, is an eigenfunction of L, so $Lu = \lambda u$ for some constant λ, show that $\lambda > 0$. [Hint: Compute $\langle Lu, u \rangle$ and integrate by parts.]

c) Find the eigenvalues λ_k and eigenfunctions $u_k(x)$ of L (remember to use the boundary conditions $u(0) = u(\pi) = 0$).

[Last revised: February 28, 2014]