Problem Set 6

DUE: In class Thurs, March 6. *Late papers will be accepted until 1:00 PM Friday.*

1. [BRETSCHER, SEC. 5.1 #26] Find the orthogonal projection P_S of $\vec{x} := \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix}$ into the subspace S of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

   SOLUTION: We are fortunate that the vectors $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. We want to find constants a and b so that

$$\vec{x} = a\vec{v}_1 + b\vec{v}_2 + \vec{w}, \tag{1}$$

   where $\vec{w}$ is orthogonal to S. Then the desired projection will be $P_S\vec{x} = a\vec{v}_1 + b\vec{v}_2$. To find the scalars a and n, take the inner product of (1) with $\vec{v}_1$ and then $\vec{v}_2$ we find

$$\langle \vec{x}, \vec{v}_1 \rangle = a\|\vec{v}_1\|^2 \qquad \text{and} \qquad \langle \vec{x}, \vec{v}_2 \rangle = b\|\vec{v}_2\|^2.$$

   Using the particular vectors in this problem, $a = 11$ and $b = -1$. Thus

$$P_S\vec{x} = 11\vec{v}_1 - \vec{v}_2 = \begin{pmatrix} 19 \\ 39 \\ 64 \end{pmatrix}$$

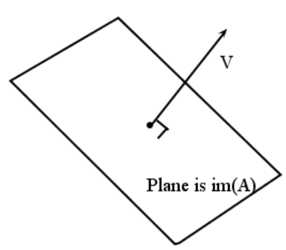
2. [BRETSCHER, SEC. 5.4 #2] Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$. Find a basis for $\ker A^*$.

   Draw a sketch illustrating the formula $(\operatorname{im} A)^\perp = \ker A^*$ in this case.

   SOLUTION: We need to solve $A^*\vec{x} = 0$, namely:

$$x_1 + x_2 + x_3 = 0, \quad x_1 + 2x_2 + 3x_3 = 0$$

   Hence we obtain that $x_2 = -2x_3$ and $x_1 = x_3$ so $\vec{v} := (1, -2, 1)$ is a basis for $\ker A^*$.



3. [Bretscher, Sec. 5.4 #16] Let $A : \mathbb{R}^k \to \mathbb{R}^n$ be an $n \times k$ matrix. Show that

$$\operatorname{rank} A^* = \operatorname{rank} A, \qquad \text{that is,} \qquad \dim(\operatorname{image} A^*) = \dim(\operatorname{image} A)$$

Solution: We will use the two formulas

$$(\operatorname{image} A)^\perp = \ker A^* \qquad \text{and} \qquad \dim(\ker A^*) + \dim(\operatorname{image} A^*) = n$$

[or the equivalent formulas interchanging the roles of A and A^*]. Since

$$\dim(\operatorname{image} A) + \dim(\operatorname{image} A)^\perp = n,$$

the first formila implies

$$n - \dim(\operatorname{image} A) = \dim(\ker A^*),$$

while the second implies

$$n - \dim(\operatorname{image} A^*) = \dim(\ker A^*)$$

The result is now clear.

4. [Bretscher, Sec. 5.2 #32] Find an orthonormal basis for the plane $x_1 + x_2 + x_3 = 0$.

Solution: Pick any point in the plane, say $\vec{v}_1 = (1, -1, 0)$. This will be the first vector in our orthogonal basis. We use the Gram-Schmidt process to extend this to an orthogonal basis for the plane.

Pick any other point in the plane, say $\vec{w}_1 := (1, 0 - 1)$. Write it as $\vec{w}_1 = a\vec{v}_1 + \vec{z}$, where $\vec{z}$ is perpendicular to $\vec{v}_1$. Note that, although unknown, $\vec{z}$ will also be in the plane since it will be a linear combination of $\vec{v}_1$ and $\vec{w}$, both of which are in the plane. As usual, by taking the inner product of both sides of $\vec{w}_1 = a\vec{v}_1 + \vec{z}$ with $\vec{v}_1$, we find

$$a = \langle \vec{w}_1, \vec{v}_1 \rangle / \|\vec{v}_1\|^2 = \frac{1}{2}.$$

Thus

$$\vec{z} = \vec{w}_1 - \frac{1}{2}\vec{v}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

is in the plane and orthogonal to $\vec{v}_1$. The vectors $\vec{v}_1$ and $\vec{z}$ are an orthogonal basis for this plane. To get an *orthonormal* basis we just make these into unit vectors

$$\vec{u}_1 := \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 := \frac{\vec{z}}{\|\vec{z}\|} = \frac{1}{\sqrt{3/2}} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$$

5. [BRETSCHER, SEC. 5.3 #10] Consider the space $\mathcal{P}_2$ of real polynomials of degree at most 2 with the inner product

$$\langle f,\, g\rangle = \tfrac{1}{2}\int_{-1}^{1} f(t)g(t)\,dt.$$

Find an orthonormal basis for all the functions in $\mathcal{P}_2$ that are orthogonal to $f(t) = t$.

SOLUTION: We have that $\{1, t, t^2\}$ is a basis for $\mathcal{P}_2$. The orthogonal complement of t has dimension 2. Because t is an odd polynomial and both 1 and t^2 are even, the elements in $\mathcal{P}_2$ that are orthogonal to t are the polynomials of the form $p(t) = a\cdot 1 + bt^2$. We want an orthonormal basis for this.

First an orthoginal basis. Let $p_1(t) = 1$. This will be the first element of our orthogonal basis. For the second we write $t^2 = a\cdot 1 + p_2(t)$, where $p_2(t)$ is orthogonal to $p_1(t)$. As usual, take the inner product of both sides of this with $p_1(t)$ to find $\langle t^2,\, 1\rangle = a\langle 1,\, 1\rangle + \langle p_2,\, 1\rangle$. Since $\|1\| = 1$ and we want $p_2 \perp 1$, this means $\langle t^2,\, 1\rangle = a\|1\|^2 + 0 = a$. But

$$\langle t^2,\, 1\rangle = \tfrac{1}{2}\int_{-1}^{1} t^2\cdot 1\,dt = \frac{1}{3}.$$

Thus $a = 1/3$ and hence $p_2(t) = t^2 - 1/3$.

To make p_1, p_2 into an orthonormal basis we compute

$$\|p_2\|^2 = \tfrac{1}{2}\int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt = \frac{4}{45}.$$

An orthonormal basis of the polynomials in $\mathcal{P}_2$ that are orthogonal to t is thus

$$e_1(t) = 1, \qquad e_2(t) = \frac{t^2 - \frac{1}{3}}{\sqrt{4/45}} = \tfrac{\sqrt{5}}{2}(3t^2 - 1).$$

6. [BRETSCHER, SEC. 5.3 #16] Consider the space $\mathcal{P}_1$ with the inner product

$$\langle f,\, g\rangle = \int_{0}^{1} f(t)g(t)\,dt.$$

a) Find an orthonormal basis for this space. [Suggestion: Let $e_1(t) = 1$ and pick $e_2(t) = a + bt$ to be orthogonal to e_1.]

SOLUTION: We let $e_1(t) = 1$ (it already has length 1). For e_2 to be orthogonal to e_1 we need , $e_2(t) = c(t - 1/2)$ for some constant c. Since $\int_0^1 (t - 1/2)^2\,dt = 1/12$, then

$$e_2(t) = \sqrt{12}(t - 1/2) = \sqrt{3}(2t - 1).$$

b) Find the linear polynomial $g(t) = a + bt$ that best approximates the polynomial $f(t) = t^2$. Thus, one wants to pick $g(t)$ so that $\|f - g\|$ is as small as possible. [Question: In an inner product space V, if you have a subspace $S \subset V$ and a vector $\vec{y} \in V$, how can you find the vector in S that is closest to $\vec{y}$?]

SOLUTION: Use the orthogonal projection on S. Since $\langle t^2, 1\rangle = 1/3$ and $\langle t^2, \sqrt{3}(2t-1)\rangle = \frac{\sqrt{3}}{6}$, then

$$g(t) = \text{proj}_S f(t) = \sum_i \langle e_i(t), f(t)\rangle e_i(t) = 1/3 + \frac{\sqrt{3}}{6}\sqrt{3}(2t-1) = -1/6 + t.$$

7. Let $f(x) := \begin{cases} 0 & \text{if } -\pi \le x \le -\pi/2 \\ 1 & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 \le x \le \pi \end{cases}$ and define $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Find the Fourier Series of $f(x)$.

SOLUTION: Using the formulas for the coefficients we have: $a_0 = \sqrt{\pi/2}$ and for $n \ge 1$,

$$a_n = \int_{-\pi/2}^{\pi/2} \frac{\cos nx}{\sqrt{\pi}}\,dx = \frac{2}{n\sqrt{\pi}}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{2}{n\sqrt{\pi}} & \text{if } n = 1, 5, 9, \ldots, \\ -\frac{2}{n\sqrt{\pi}} & \text{if } n = 3, 7, 11, \ldots \end{cases}.$$

Similarly, since $\sin nx$ is an odd function,

$$b_n = \frac{1}{\sqrt{\pi}} \int_{-\pi/2}^{\pi/2} \sin nx\,dx = 0,$$

Hence,

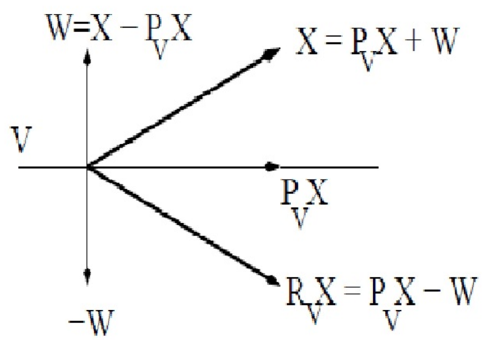
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \cdots\right]$$

8. [BRETSCHER, SEC. 5.1 #37] Consider a plane V in $\mathbb{R}^3$ with orthonormal basis $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x}$ be a vector in $\mathbb{R}^3$. Find a formula for the orthogonal reflection $R\vec{x}$ of $\vec{x}$ across the plane V. Your answer will involve $P_V\vec{x}$, the orthogonal projection of $\vec{x}$ into the plane V. [Suggestion: Use that $(I - P_V)\vec{x}$ is the component of $\vec{x}$ that is orthogonal to V. In a reflection, this is the part of $\vec{x}$ that is flipped.]

SOLUTION: The key is a picture (first try it in $\mathbb{R}^2$ where V is a line through the origin). Let $P_V\vec{x}$ be the orthogonal projection of $\vec{x}$ into the plane V. Then $\vec{w} := P_{V^\perp}\vec{x} = \vec{x} - P_V\vec{x}$ is the projection of $\vec{x}$ orthogonal to V. From the picture, to get the reflection, replace $\vec{w}$ by $-\vec{w}$

Thus, since $\vec{x} = P_V\vec{x} + \vec{w}$, then

$$R_V\vec{x} = P_V\vec{x} - \vec{w} = P_V\vec{x} - (\vec{x} - P_V\vec{x}) = 2P_V\vec{x} - \vec{x}.$$



In summary, orthogonal projections and reflections for a subspace V are related by the simple formula $R_V = 2P_V - I$.

Note that if you know an orthonormal basis for V the orthogonal projection, $P_V\vec{x}$, is easy to compute All of this is very general. In this problem $\vec{u}_1$ and $\vec{u}_2$ are an orthonormal basis for the subspace V, so

$$P_V\vec{x} = \langle \vec{x},\, \vec{u}_1\rangle\vec{u}_1 + \langle \vec{x},\, \vec{u}_2\rangle\vec{u}_2.$$

Consequently,

$$R_V\vec{x} = 2(\langle \vec{x},\, \vec{u}_1\rangle\vec{u}_1 + \langle \vec{x},\, \vec{u}_2\rangle\vec{u}_2) - \vec{x}.$$

9. Let V be a linear space with an inner product and $P : V \to V$ a linear map. P is called a *projection* if $P^2 = P$. Let $Q := I - P$.

a) Show that $Q^2 = Q$, so Q is also a projection.
Show that the image of P is the kernel of Q.

SOLUTION: $Q^2 = I - PI - IP + P^2 = I - P - P + P = I - P = Q$.
We need to show that $\operatorname{im} P \subset \ker Q$ and $\ker Q \subset \operatorname{im} P$. Say $x \in \operatorname{im} P$, then $x = Py$ for some y. Thus

$$(I - P)x = Ix - Px = Ix - P^2y = Ix - Py = x - x = 0.$$

Conversely, say $y \in \ker Q$, then $y = Iy = Py$ so $y \in \operatorname{im} P$.

b) A projection P is called an *orthogonal projection* if the image of P is orthogonal to the kernel of P. If $P = P^*$, show that P is an orthogonal projection.

SOLUTION: Let $x \in \ker Q = \operatorname{im} P$ and $y \in \ker P$. Since $x = Px$ and $Py = 0$, then $\langle x,\, y\rangle = \langle Px,\, y\rangle = \langle x,\, P^*y\rangle = \langle x,\, Py\rangle = 0$.

c) Conversely, if P is an orthogonal projection, show that $P = P^*$.

SOLUTION: We will show that $\langle Px,\, y\rangle = \langle x,\, Py\rangle$ for all x and y. Write $x = Px + (I - P)x = x_1 + x_2$. Note that $x_1 \in \operatorname{im}(P)$ and $x_2 \in \ker(P)$. Similarly write $y = Py + (I - P)y = y_1 + y_2$. By assumption the image and kernel of P are orthogonal, so x_1 and y_2 are orthogonal, as are x_2 and y_1. The following computation completes the proof.

$$\langle Px,\, y\rangle = \langle x_1,\, y_1 + y_2\rangle = \langle x_1,\, y_1\rangle \qquad \text{and} \qquad \langle x,\, Py\rangle = \langle x_1 + x_2,\, y_1\rangle = \langle x_1,\, y_1\rangle.$$

ALTERNATE: Since the image and kernel of P are orthogonal, then $\langle (I-P)x,\, Py\rangle = 0$ for all x and y. Thus,

$$\langle x,\, Py\rangle = \langle Px,\, Py\rangle = \langle x,\, P^*Py\rangle$$

for all x and y. This implies that $P = P^*P$. Since P^*P is self-adjoint, this shows that P is self-adjoint.

10. Let A be a real matrix, not necessarily square.

 a) If A is onto, show that A^* is one-to-one.

 SOLUTION: Since $\operatorname{im} A^{\perp} = \ker A^*$, thus $\ker A = 0$.

 b) If A is one-to-one, show that A^* is onto.

 SOLUTION: Similarly, $\operatorname{im} A^{*\perp} = \ker A$.

11. Let A be a real matrix, not necessarily square.

 a) Show that both A^*A and AA^* are self-adjoint.

 SOLUTION: Using $(AB)^* = B^*A^*$ and $(A^*)^* = A$, this is easy.

 The example $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is illuminating. Here

$$A^*A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad AA^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

 b) Show that $\ker A = \ker A^*A$. [HINT: Show separately that $\ker A \subset \ker A^*A$ and $\ker A \supset \ker A^*A$. The identity $\langle \vec{x},\, A^*A\vec{x}\rangle = \langle A\vec{x},\, A\vec{x}\rangle$ is useful.]

 SOLUTION: If $\vec{x} \in \ker A$, then $A\vec{x} = 0$ so $A^*A\vec{x} = A^*0 = 0$. Thus $\vec{x} \in \ker A^*A$. In other words, $\ker A \subset \ker A^*A$.

 Conversely, if $\vec{x} \in \ker A^*A$, then $A^*A\vec{x} = 0$ so

$$0 = \langle \vec{x},\, A^*A\vec{x}\rangle = \langle A\vec{x},\, A\vec{x}\rangle = \|A\vec{x}\|^2.$$

 Consequently $A\vec{x} = 0$, that is, $\vec{x} \in \ker A$. This proves that $\ker A^*A \subset \ker A$.

 c) If A is one-to-one, show that A^*A is invertible

 SOLUTION: From part (b) the square matrix A^*A is 1-1, hence it is invertible.

d) If A is onto, show that AA^* is invertible.

SOLUTION: From exercise 10, part (a) we have that A^* is 1-1. Therefore as in part (c), AA^* is 1-1 so the square matrix AA^* is invertible.

12. [This question is now a bonus question (see below).]

Quadratic Polynomials Using Inner Products

If A is a real symmetric matrix (so it is self-adjoint), then $Q(\vec{x}) := \langle \vec{x},\, A\vec{x} \rangle$ is a quadratic polynomial. Given a quadratic polynomial, it is easy to find the (unique) symmetric symmentic matrix A. Here is an example. Say $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2$ To find A, note that $-8x_1x_2 = -4x_1x_2 - 4x_2x_2$ so we can rewite Q as

$$Q(\vec{x}) := 3x_1^2 - 4x_1x_2 - 4x_2x_1 - 5x_2^2.$$

If we let

$$A := \begin{pmatrix} 3 & -4 \\ -4 & -5 \end{pmatrix} \qquad \text{[Note } A \text{ is a symmetric matrix]},$$

then it is easy to verify that $Q(\vec{x}) = \langle \vec{x},\, A\vec{x} \rangle$. In the remaining problems we will use this to help work with quadratic polynomials.

13. In each of these find a 3×3 *symmetric* matrix A so that $Q(\vec{x}) = \langle \vec{x},\, A\vec{x} \rangle$.

a) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 + x_3^2$.

SOLUTION: $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3 + x_3^2$.

SOLUTION: $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -5 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}$

c) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3$.

SOLUTION: $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -5 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$

14. [LOWER ORDER TERMS AND COMPLETING THE SQUARE] Which is simpler:

$$z = x_1^2 + 4x_2^2 - 2x_1 + 4x_2 + 2 \qquad \text{or} \qquad z = y_1^2 + 4y_2^2\ ?$$

If we let $y_1 = x_1 - 1$ and $y_2 = x_2 + 1/2$, they are essentially the same. All we did was translate the origin to $(1,\, -1/2)$.

The point of this problem is to generalize this to quadratic polynomials in several variables. Let

$$\begin{aligned} Q(\vec{x}) &= \sum a_{ij}x_ix_j + 2\sum b_ix_i + c \\ &= \langle \vec{x},\, A\vec{x}\rangle + 2\langle b,\, \vec{x}\rangle + c \end{aligned}$$

be a real quadratic polynomial so $\vec{x} = (x_1, \dots, x_n)$, $\vec{b} = (b_1, \dots, b_n)$ are real vectors and $A = (a_{ij})$ is a real symmetric $n \times n$ matrix.

In the case $n = 1$, $Q(x) = ax^2 + 2bx + c$ which is clearly simpler in the special case $b = 0$. In this case, if $a \neq 0$, by completing the square we find

$$Q(x) = a\,(x + b/a)^2 + c - 2b^2/a = ay^2 + \gamma,$$

where we let $y = x - b/a$ and $\gamma = c - b^2/a$. Thus, by translating the origin: $x = y + b/a$ we can eliminate the linear term in the quadatratic polynomial – so it becomes simpler.

a) Similarly, for any dimension n, if A is invertible, using the above as a model, show there is a change of variables $\vec{y} == \vec{x} - \vec{v}$ (this is a translation by the vector $\vec{v}$) so that in the new $\vec{y}$ variables Q hasthe form

$$\hat{Q}(\vec{y}) := Q(\vec{y} + \vec{v}) = \langle \vec{y},\, A\vec{y}\rangle + \gamma \quad \text{that is,} \quad \hat{Q}(\vec{y}) = \sum a_{ij}y_iy_j + \gamma,$$

where γ involves A, b, and c – but no terms that are linear in $\vec{y}$. [In the case $n = 1$, which you should try *first*, this means using a change of variables $y = x - v$ to change the polynomial $ax^2 + 2bx + c$ to the simpler $ay^2 + \gamma$.]

SOLUTIONS: First the case $n = 1$ again. Then $Q(x) = Ax^2 + 2bx + c$ so

$$\begin{aligned} Q(x) = Q(y + v) &= A(y + v)^2 + 2b(y + v) + c \\ &= Ay^2 + (2Av + 2b)y + Av^2 + 2bv + c. \end{aligned}$$

To kill the linear term, pick v so that $2Av + 2b = 0$, that is, $v = -b/A$. Then $Q(x) = Ay^2 + \gamma$, where

$$\gamma = Ab^2/A^2 - 2b^2/A + c = -b^2/A + c.$$

Next, the case of arbitrary n. It should now feel routine. We are trying the change of variables $\vec{x} == \vec{y} - \vec{v}$ with the thought of picking $\vec{v}$ to simplify the result. The following should be a straightforward computation (the third line uses $A = A^*$):

$$\begin{aligned} Q(\vec{x}) &= Q(\vec{y} + \vec{v}) = \langle \vec{y} + \vec{v},\, A(\vec{y} + \vec{v})\rangle + \langle \vec{b},\, \vec{y} + \vec{v}\rangle + c \\ &= \langle \vec{y},\, A\vec{y}\rangle + \langle \vec{y},\, A\vec{v}\rangle + \langle \vec{v},\, A\vec{y}\rangle + \langle \vec{v},\, A\vec{v}\rangle + 2\langle \vec{b},\, \vec{y}\rangle + 2\langle \vec{b},\, \vec{v}\rangle + c \\ &= \langle \vec{y},\, A\vec{y}\rangle + \langle 2A\vec{v} + 2\vec{b},\, \vec{y}\rangle + \langle \vec{v},\, A\vec{v}\rangle + 2\langle \vec{b},\, \vec{v}\rangle + c. \end{aligned}$$

The term that is linear in $\vec{y}$ will vanish if we pick $\vec{v}$ so that $2A\vec{v} + 2\vec{b} = 0$, that is, $\vec{v} = -A^{-1}\vec{b}$. Then

$$Q(\vec{x}) = \langle \vec{y},\, A\vec{y} \rangle + \gamma$$

where

$$\gamma = \langle A^{-1}\vec{b},\, \vec{b} \rangle - 2\langle \vec{b},\, A^{-1}\vec{b} \rangle + c = -\langle \vec{b},\, A^{-1}\vec{b} \rangle + c.$$

This agrees with what we found in the special case $n = 1$.

b) As an example, apply this to $Q(\vec{x}) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4$.

SOLUTION: Here $Q(\vec{x}) = \langle \vec{x},\, A\vec{x} \rangle + 2\langle \vec{b},\, \vec{x} \rangle + c$, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}$, and $c = -4$. Thus $A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$ so $\vec{v} = -A^{-1}\vec{b} = \begin{pmatrix} 3/2 \\ -3 \end{pmatrix}$.

15. For $\vec{x} \in \mathbb{R}^n$ let $Q(\vec{x}) := \langle \vec{x},\, A\vec{x} \rangle$, where A is a real symmetric matrix. We say that A is *positive definite* if $Q(\vec{x}) > 0$ for all $\vec{x} \neq 0$, *negative definite* if $Q(\vec{x}) < 0$ for all $\vec{x} \neq 0$, and *indefinite* if $Q(\vec{x}) > 0$ for some $\vec{x}$ but $Q(\vec{x}) < 0$ for some other $\vec{x}$.

a) In the special case $n = 2$ give (simple!) examples of matrices A that are positive definite, negative definite, and indefinite.

SOLUTION: Several examples. Begin with the polynomial, not the matrix.
positive definite: If $\langle \vec{x},\, A\vec{x} \rangle = x_1^2 + x_2^2$ then A is the identity matrix I, and $\langle \vec{x},\, A\vec{x} \rangle = 2x_1^2 + 3x_2^2$ so $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$.
negative definite: For $\langle \vec{x},\, A\vec{x} \rangle = -x_1^2 - x_2^2$, the matrix is $-I$ while for $\langle \vec{x},\, A\vec{x} \rangle = -2x_1^2 - 3x_2^2$, the matrix is $\left(\begin{smallmatrix} -2 & 0 \\ 0 & -3 \end{smallmatrix}\right)$.
indefinite: For $\langle \vec{x},\, A\vec{x} \rangle = x_1^2 - x_2^2$ the matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ while for $\langle \vec{x},\, A\vec{x} \rangle = -2x_1^2 + 5x_2^2$ the matrix is $\left(\begin{smallmatrix} -2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$.
NOTE: If $\langle \vec{x},\, A\vec{x} \rangle = 3x_2^2$, the matrix is $A := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ is *not* positive definite, it is *positive semi-definite*, that is, $\langle \vec{x},\, A\vec{x} \rangle \geq 0$ for all $\vec{x}$ but $\langle \vec{x},\, A\vec{x} \rangle = 0$ for some $\vec{x} \neq 0$.

b) In the special case where A is an invertible *diagonal* matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

under what conditions is $Q(\vec{x})$ positive definite, negative definite, and indefinite? [REMARK: We will see that the general case can *always* be reduced to this special case where A is diagonal.]

SOLUTION: Key step: here

$$\langle \vec{x},\, A\vec{x} \rangle = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2.$$

If we let $\vec{x} = (0, 1, 0, \dots, 0)$, clearly $\langle \vec{x}, A\vec{x} \rangle = \lambda_2$ so if A is positive definite, then $\lambda_2 > 0$. Similarly, if A is positive definite, then all the λ_j are positive.

Conversely, if all the λ_J are positive, it is clear that A is positive definite.

By the same reasoning, A is negative definite if (and only if) all the $\lambda_j < 0$, and indefinite if at least one λ_j is positive and another is negative.

NOTE: the assumption "A is invertible" implies that none of the λ_j are zero.

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let $\mathcal{S} := \{u(x) \in C^2[0, \pi]$ with $u(0) = u(\pi) = 0\}$ and let $Lu := -u''(x)$. Use the inner product $\langle u, v \rangle = \int_0^\pi u(x)v(x)\,dx$.

a) If u and v are in $\mathcal{S}$, show that $\langle Lu, v \rangle = \langle u, Lv \rangle$. This shows that L is self-adjoint on this space of functions. [HINT: Integrate by parts.]

SOLUTION: Using integration by parts you obtain $\langle Lu, v \rangle = \int_0^\pi u'v'\,dx$ and $\langle v, Lu \rangle = \langle Lu, v \rangle = \int_0^\pi u'v'\,dx$.

b) If $u(x) \in \mathcal{S}$, $u \not\equiv 0$, is an eigenfunction of L, so $Lu = \lambda u$ for some constant λ, show that $\lambda > 0$. [HINT: Compute $\langle Lu, u \rangle$ and integrate by parts.]

SOLUTION: If $\lambda = 0$ then u solves $u'' = 0$ and get $u \notin \mathcal{S}$ so we have a contradiction. Hence $\lambda \neq 0$. Now, $\lambda\langle u, u \rangle = \langle Lu, u \rangle = \int_0^\pi (u')^2\,dx \geq 0$ Hence $\lambda \geq 0$. Thus $\lambda > 0$.

c) Find the eigenvalues λ_k and eigenfunctions $u_k(x)$ of L (remember to use the boundary conditions $u(0) = u(\pi) = 0$).

SOLUTION: For this part see to the notes:

`http://hans.math.upenn.edu/~kazdan/312S13/notes/Lu=-DDu.pdf` .

B-2 Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map that is onto but not one-to-one. Say X_1 is a solution of $AX = Y$. Is there a "best" possible solution? What can one say? Think about this before reading the next paragraph.

a) Show that AA^* is invertible so there is exactly one solution V of $AA^*V = Y$. Thus the vector $X_2 := A^*V$ is also a solution of $AX = Y$.

SOLUTION: Since A is onto we have that A^* is one-to-one, namely $\ker A^* = \{0\}$ and hence that the square matrix AA^* is invertible. [This is the same as Problem 11d) above.]

b) Show that if X_1 is *any* solution of $AX = Y$, then X_2 is closer to the origin, that is, $\|X_2\| \leq \|X_1\|$. In other words, X_2 is the solution that is closest to the origin. [HINT: the general solution of $AX = Y$ is $X = X_2 + Z$ where $Z \in \ker A$.]

SOLUTION: We have that $X_2 = A^*V \in \operatorname{im} A^* = (\ker A)^\perp$ and $X_1 = X_2 + Z$ for some $Z \in \ker A$, hence Z and X_2 are orthogonal. Then by the Pythagorean theorem we have that

$$\|X_1\|^2 = \|X_2 + Z\|^2 = \|X_2\|^2 + \|Z\|^2 \geq \|X_2\|^2.$$

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