Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 7

DUE: In class Thurs, March 20. *Late papers will be accepted until 1:00 PM Friday.*

For the week after Spring Break, please read Chapter 6 on determinants and, as a review ofMath 240, Chapter 7.1 and 7.2.

1. For real $c > 0$, $c \neq 1$, and distinct points $\vec{p}$ and $\vec{q}$ in $\mathbb{R}^n$, consider the points $\vec{x} \in \mathbb{R}^n$ that satisfy
$$\|\vec{x} - \vec{p}\| = c\|\vec{x} - \vec{q}\|.$$
Show that these points lie on some sphere: $\|\vec{x} - \vec{x}_0\| = r$, with center at $\vec{x}_0$ and radius r. This problem is to find the center and radius of the sphere in terms of $\vec{p}$, $\vec{q}$, and c.

   What happens in the special case $c = 1$?

2. [BRETSCHER, SEC. 5.5 #24]. Using the inner product $\langle f,\, g\rangle := \int_0^1 f(x)g(x)\,dx$, for certain polynomials $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ say we are given the following table of inner products:

$\langle\ ,\ \rangle$	$\mathbf{f}$	$\mathbf{g}$	$\mathbf{h}$
$\mathbf{f}$	4	0	8
$\mathbf{g}$	0	1	3
$\mathbf{h}$	8	3	50

   For example, $\langle \mathbf{g}, \mathbf{h}\rangle = \langle \mathbf{h}, \mathbf{g}\rangle = 3$. Let E be the span of $\mathbf{f}$ and $\mathbf{g}$.

   a) Compute $\langle \mathbf{f}, \mathbf{g} + \mathbf{h}\rangle$.

   b) Compute $\|\mathbf{g} + \mathbf{h}\|$.

   c) Find $\text{proj}_E \mathbf{h}$. [Express your solution as linear combinations of $\mathbf{f}$ and $\mathbf{g}$.]

   d) Find an orthonormal basis of the span of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ [Express your results as linear combinations of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$.]

3. Lat $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the conditions on the coefficients a, b, c, and d so that A is an orthogonal matrix.

4. [See Homework 4, Problem 13] Say $X(t)$ is a solution of the differential equation $\dfrac{dX}{dt} = AX$, where A is an anti-symmetric matrix. Show that $\|X(t)\| = \text{constant}$. [REMARK: A special case is that $X(t) := \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ satisfies $X' = AX$ with $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ so this problem gives another proof that $\cos^2 t + \sin^2 t = 1$.]

5. [Bretscher, Sec. 5.4 #20] Using pencil and paper, find the least-squares solution to $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

6. Use the Method of Least Squares to find the parabola $y = ax^2 + b$ that best fits the following data given by the following four points (x_j, y_j), $j = 1, \ldots, 4$:

$$(-2, 4), \qquad (-1, 3), \qquad (0, 1), \qquad (2, 0).$$

Ideally, you'd like to pick the coefficients a and b so that the four equations $ax_j^2 + b = y_j$, $j = 1, \ldots, 4$ are all satisfied. Since this probably can't be done, one uses least squares to find the best possible a and b.

7. The water level in the North Sea is mainly determined by the so-called M2 tide, whose period is about 12 hours see [`https://en.wikipedia.org/wiki/Tide`]. The height $H(t)$ thus roughly has the form

$$H(t) = c + a\sin(2\pi t/12) + b\cos(2\pi t/12),$$

where time t is measured in hours (note $\sin(2\pi t/12$ and $\cos(2\pi t/12)$ are periodic with period 12 hours). Say one has the following measurements:

t (hours)	0	2	4	6	8	10
$H(t)$ (meters)	1.0	1.6	1.4	0.6	0.2	0.8

Use the method of least squares to find the constants a, b, and c in $H(t)$ for this data. [Although one can certainly solve this "by hand," you may prefer to use computer software.]

8. Some experimental data $(x_i,\ y_i)$ is believed to fit a curve of the form

$$y = \frac{1 + x}{a + bx^2},$$

where the parameters a and b are to be determined from the data. The method of least squares does not apply directly to this since the parameters a and b do not appear linearly. Show how to find a modified equation to which the method of least squares does apply.

9. **Plotting graphs** This problem concerns the straight line in the plane that passes through the two points $(4, 0)$ and $(0, 2)$ (draw a sketch).

a) If the horizontal axis is x and the vertical axis y, what is the equation for y as a function of x?

b) If the horizontal axis is $\log x$ and the vertical axis y, what is the equation for y as a function of x?

c) If the horizontal axis is x and the vertical axis $\log y$, what is the equation for y as a function of x?

d) If the horizontal axis is $\log x$ and the vertical axis $\log y$, what is the equation for y as a function of x?

10. Say $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with the property that $A^2 - 3A + 2I = 0$. If $\vec{v} \neq 0$ is an eigenvector of A with eigenvalue λ, that is, $A\vec{v} = \lambda\vec{v}$, what are the possible values of λ?

11. Let R be a (real) 3×3 orthogonal matrix.

    a) Show that the eigenvalues, λ, which may be complex, all have absolute value 1.

    b) If $\det R = 1$ show that $\lambda = 1$ is one of the eigenvalues of R and that if $R \neq I$, no other eigenvalue can be 1.

    For the remainder of this problem assume $\det R = 1$ *and* $R \neq I$.

    c) Let N be an eigenvector corresponding to $\lambda = 1$ and let Q be the plane of all vectors orthogonal to N. Show that R maps Q to Q.

    d) Why does this show that R is a rotation of the plane Q with N as the axis of rotation?

12. Let A be a 3×3 matrix with eigenvalues λ_1, λ_2, λ_3 and corresponding linearly independent eigenvectors V_1, V_2, V_3 which we can therefore use as a basis.

    a) If $X = aV_1 + bV_2 + cV_3$, compute AX, A^2X, and $A^{35}X$ in terms of λ_1, λ_2, λ_3, V_1, V_2, V_3, a, b and c (only).

    b) If $\lambda_1 = 1$, $|\lambda_2| < 1$, and $|\lambda_3| < 1$, compute $\lim_{k\to\infty} A^k X$. Explain your reasoning clearly.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Let $P_1,\ P_2, \ldots,\ P_k$ be k points (think of them as *data*) in $\mathbb{R}^3$ and let $\mathcal{S}$ be the plane

$$\mathcal{S} := \left\{X \in \mathbb{R}^3\ :\ \langle X,\, N\rangle = c\right\},$$

where $N \neq 0$ is a unit vector normal to the plane and c is a real constant. [Note: This plane $\mathcal{S}$ is only a subspace if $c = 0$. The subspace parallel to this plane satisfies $\langle X,\, N\rangle = 0$ so, writing more carefully, N is orthogonal to $\mathcal{S}$ only when $c = 0$. Indeed, a preliminary first step (see below), is to make a change of coordinates to reduce to this special case.]

This problem outlines how to find the plane that *best approximates the data points* in the sense that it minimizes the function

$$Q(N, c) := \sum_{j=1}^{k} \text{distance}\,(P_j, \mathcal{S})^2.$$

Determining this plane means finding N and c.

a) Show that for a given point P, then

$$\text{distance}\,(P, \mathcal{S}) = |\langle P - X,\, N\rangle| = |\langle P,\, N\rangle - c\,|,$$

where X is any point in $\mathcal{S}$

b) First do the special case where the center of mass $\overline{P} := \frac{1}{k}\sum_{j=1}^{k} P_j$ is at the origin, so $\overline{P} = 0$. Show that for any P, then $\langle P,\, N\rangle^2 = \langle N,\, PP^*N\rangle$. Here view P as a column vector so PP^* is a 3×3 matrix.

Use this to observe that the desired plane $\mathcal{S}$ is determined by letting N be an eigenvector of the matrix

$$A := \sum_{j=1}^{k} P_j P_j^T$$

corresponding to it's lowest eigenvalue. What is c in this case?

c) Reduce the general case to the previous case by letting $V_j = P_j - \overline{P}$.

d) Find the equation of the line $ax + by = c$ that, in the above sense, best fits the data points $(-1, 3)$, $(0, 1)$, $(1, -1)$, $(2, -3)$.

e) Let $P_j := (p_{j1}, \ldots, p_{j3})$, $j = 1, \ldots, k$ be the coordinates of the j^{th} data point and $Z_\ell := (p_{1\ell}, \ldots, p_{k\ell})$, $\ell = 1, \ldots, 3$ be the vector of ℓ^{th} coordinates. If a_{ij} is the ij element of A, show that $a_{ij} = \langle Z_i,\, Z_j\rangle$. Note that this exhibits A as a *Gram matrix* .

[Last revised: March 18, 2014]