Math 312, Spring 2014 Jerry L. Kazdan

Problem Set 8

DUE: In class Thurs, March 27. *Late papers will be accepted until 1:00 PM Friday.*

For the coming week, please read all of Chapter 7 and the first few sections of Chapter 8.

> **Reminder:** Exam 2 is on Tuesday, April 1, 9:00–10:20. No books or calculators but you may always use one 3" × 5" card with handwritten notes on both sides.

1. [REVIEW:] Find the eigenvalues and eigenvectors of $A := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

   Find an othogonal matrix R so that $R^{-1}AR$ is a diagonal matrix.

2. Multinational companies in the Americas, Asia, and Europe have assets of $4 trillion. At the start, $2 trillion are in the Americas and $2 trillion are in Europe. Each year 1/2 of the Americas money stays home and 1/4 goes to each of Asia and Europe. For Asia and Europe, 1/2 stays home and 1/2 is sent to the Americas.
   a) Let C_k be the column vector with the assets of the Americas, Asia, and Europe at the beginning of year k. Find the transition matrix T that gives the amount in year $k+1$: $C_{k+1} = TC_k$
   b) Find the eigenvalues and eigenvectors of T.
   c) Find the limiting distribution of the $4 trillion as the world ends
   d) Find the distribution of the $4 trillion at year k.

3. A certain plant species has either red, pink, or white flowers, depending on its genotype. If you cross a pink plant with any other plant, the probability distribution of the offsprings are prescribed by the transition matrix

$$T := \begin{pmatrix} .5 & .25 & 0 \\ .5 & .5 & .5 \\ 0 & .25 & .5 \end{pmatrix}.$$

   The first column of T means that if you cross a red with a pink, then 50% of the time you'll get a white and 50% a pink. The second column gives the result if you cross a pink with a pink, while the third column concerns crossing a white with a pink.

   In the long run, if you continue crossing the offsprings with only pink plants, what percentage of the three types of flowers would you expect to see in your garden?

4. Let T be the transition matrix of a Markov process.

a) If P is a probability vector, show that TP is also a probability vector.

b) If T is the transition matrix of a *regular* Markov process (so for some k all if entries of T^k are positive), we know there is a probability vector P_∞ so that if P_0 is any initial probability vector, then $\lim_{k\to\infty} T^k P_0 = P_\infty$. Show that the matrix $lim_{k\to\infty} T^k$ has all of its columns equal to P_∞.

5. Let A and B be $n \times n$ matrices that can both be diagonalized by the *same* matrix S, so $A = SD_1S^{-1}$ and $B = SD_2S^{-1}$, where D_1 and D_2 are both diagonal matrices. Show that $AB = BA$.

6. [Bretscher Sec. 7.2 #32] Consider the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & 3 & 0 \end{pmatrix}$, where c is an arbitrary real number. For which values of c does A have three real eigenvalues? [Suggestion: Graph the characteristic polynomial.]

7. Find the eigenvalues and eigenvectors of $B := \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

8. A certain real 4×4 matrix A has $\lambda_1 = 2 - 5i$ and $\lambda_2 = 1 + 2i$ as eigenvalues. What are the other two eigenvalues? Can you diagonalize A? Why or why not?

9. [Bretscher Sec. 7.3 #40, 41, 44] Let A and B be $n \times n$ matrices.

   a) Show that $\text{trace}(AB) = \text{trace}(BA)$.

   b) Use this to give another proof that if A and C are similar, then $\text{trace}(A) = \text{trace}(C)$.

   c) Are there $n \times n$ matrices so that $AB - BA = I$?

10. [Bretscher, 5th ed Sec. 7.5 #32(a)] Consider the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$, where $A := \begin{pmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.4 \end{pmatrix}$, perhaps modeling the way people search a mini-web. [Note that A is the transition matrix of a Markov Chain.]

    Using technology (say the Maple example:

    `http://hans.math.upenn.edu/~kazdan/312S14/MarkovChain2014A.mw`),

    compute high powers of A, say A^6, A^{16} and A^{32}), and make a conjecture about $\lim_{t\to\infty} A^t$.

11. If M is a square matrix, define e^M by the power series

$$e^M = I + M + \frac{M^2}{2!} + \cdots + \frac{M^k}{k!} + \cdots .$$

We will take the convergence of this series for granted (it is not difficult – but we skip this).

a) If $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, compute e^A.

b) If $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, compute J^2, J^3, J^4, and J^5.
For real t, use the above to show that

$$e^{\left(\begin{smallmatrix} 0 & -t \\ t & 0 \end{smallmatrix}\right)} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(The matrix on the right is a rotation of $\mathbb{R}^2$ through the angle t).

c) If the square matrix A is similar to B. say $A = SBS^{-1}$, show that $e^A = Se^BS^{-1}$.

d) $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, compute e^A. [Hint: See Problem 1 above.]

e) If A does not depend on t, show that $\dfrac{de^{At}}{dt} = Ae^{At}$.

f) If A is a diagonalizable constant square matrix, show that the solution of $\dfrac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \vec{b}$ is $\vec{x}(t) = e^{At}\vec{b}$.

[Last revised: March 27, 2014]