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Math 312
March 5, 1998

Hour Exam 1

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12:00 –1:20

DIRECTIONS: This exam has three parts. Part A has 4 True-False questions, Part B has 3 short answer questions, and Part C has 6 traditional problems. Each problem is worth 10 points. 130 points total.

To receive full credit your solution must be clear and correct. No fuzzy reasoning. Partial credit will *only* be given for the problems in Part C. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3×5 card with notes. Please [box] your answers where appropriate.

NOTE: *To be fair to everyone, those who submit their exam paper late* (after 1:20) *will be "charged" 5 points for every 2 additional minutes.*

PART A. Four True-False questions. 10 points each, so 2 points for each item.

Score	
A	
B	
C-1	
C-2	
C-3	
C-4	
C-5	
C-6	
Bonus	
Total	

A-1. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$.

T F If $n = k$ there is always *at most one* solution.

T F If $n > k$ you can *always* solve $AX = Y$.

T F If $n > k$ the nullspace of A has dimension greater than zero.

T F If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

T F If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

A-2. Given two $n \times n$ matrices A and B with $AB = 0$, which of the following assertions *must* be true?

T F $BA = 0$

T F Either $A = 0$ or $B = 0$ (or both).

T F If $\det A = -3$, then $B = 0$.

T F If B is invertible then $A = 0$.

T F There is a vector $V \neq 0$ such that $BAV = 0$.

A-3. Circle **T** for each of the following sets that are linear spaces.

T F $\{X = (x_1, x_2, x_3)$ in R^3 with the property $x_1 - 2x_3 = 0\}$

T F The set of solutions x to the system $Ax = 0$, where A is an $m \times n$ matrix.

T F The set of 2×2 matrices A with $\det(A) = 0$.

T F The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\, dx = 0$.

T F The set of solutions $y = y(t)$ of the differential equation $y'' + y' + y = 0$.

A-4. Circle **T** for each of the following sets of vectors that are bases for R^2.

T F $\{(0, 1),\ (1, 1)\}$

T F $\{(1, 0),\ (0, 1),\ (1, 1)\}$

T F $\{(1, 0),\ (-1, 0\}$

T F $\{(1, 1),\ (1, -1)\}$

T F $\{((1, 1),\ (2, 2)\}$

PART B. Three short-answer questions. 10 points each. Partial credit will rarely be given. Please box your answers where appropriate.

B-1. Let A be an invertible square matrix. Say there is a vector V with the property that $AV = 7V$. Compute A^2V and $A^{-1}V$.

B-2. Let A be an $n \times n$ matrix. Which of the following statements are equivalent to "the matrix A is invertible"?

(a) The columns of A are linearly independent.

(b) The linear transformation $T_A : R^n \to R^n$ defined by A is 1-1.

(c) The rank of A is n.

(A) a and b only (B) b and c only (C) a and c only (D) a, b and c
(E) a only (F) None

B-3. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

b) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.

c) Find *two* solutions of $AX = \mathbf{Y}_1$.

d) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

e) If A is a square matrix, then $\det A = ?$

PART C. Six problems. 10 points each, Please box your answers where appropriate.

C-1. A linear transformation $T : R^3 \to R^3$ first rotates the xy-plane by $+90°$ (leaving the z-axis fixed), followed by an orthogonal projection onto the yz-plane. Find the standard matrix representation for T.

C-2. Consider the system of equations $\left.\begin{aligned} x+y-\ z &= a \\ x-y+2z &= b \\ 3x+y\qquad &= c \end{aligned}\right\}$

a) Find the general solution of the homogeneous equation.

b) Observe that $x=1,\ y=1,\ z=1$ is a particular solution of the inhomogeneous equations when $a=1,\ b=2$, and $c=4$. Find the most general solution of these inhomogeneous equations.

c) If $a=1,\ b=2$, and $c=3$, show these equations have *no* solution.

d) If $a=0$, $b=0$, $c=1$, show the equations have *no* solution. [Note: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$].

e) Let A be the matrix associated with these equations, $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$. Compute $\det A$ using any method (one method involves no computation at all).

C-3. Let C and B be square matrices with C invertible.

a) Show that $(CBC^{-1})^2 = C(B^2)C^{-1}$

b) If B is also invertible, is it true that $(CBC^{-1})^{-2} = C(B^{-2})C^{-1}$? Why?

C-4. Let λ be a scalar parameter and consider the system $\left.\begin{aligned} x + \lambda y = 0 \\ \lambda x + 4y = 0 \end{aligned}\right\}.$

(a) Use row reduction to determine the values of λ for which the system has infinitely many solutions.

(b) For each of the values of λ found in (a), sketch the set of solutions in the xy-plane.

(c) Let $A = \left(\begin{smallmatrix} 1 & \lambda \\ \lambda & 4 \end{smallmatrix}\right)$. For which values of λ is the nullspace of A *not* just the zero vector?

C-5. Let the invertible square matrix A have the property that its inverse equals its transpose. Show that $\det A = \pm 1$.

C-6. Answer each of the following questions *with justification*; that is, give a proof if the statement is true or provide an example to show that it is not. In each case your answers should be brief.

(a) Suppose that u, v and w are vectors in a vector space V and $T : V \to W$ is a linear map. If u, v and w are linearly dependent, is it true that $T(u)$, $T(v)$ and $T(w)$ are linearly dependent?

(b) If $T : R^6 \to R^4$ is a linear map is it possible that the nullspace of T is one dimensional?

Bonus Problem. Let $A : R^3 \to R^2$ and $B : R^2 \to R^3$, so $BA : R^3 \to R^3$. Show that BA can *not* be invertible.