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Math 312
April 23, 1998

Hour Exam 2

Jerry L. Kazdan
12:00 –1:20

Directions: This exam has three parts. Part A has 12 True-False questions (5 points each), Part B has 5 short answer questions (10 points each), and Part C has 4 traditional problems (20 points each). Thus, 190 points total.

To receive full credit your solution must be clear and correct. No fuzzy reasoning. Partial credit will *only* be given for the problems in Part C. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3×5 card with notes.

> Note: *To be fair to everyone, those who submit their exam paper late* (after 1:20) *will be "charged" 5 points for every 2 additional minutes.*

Part A. *Circle* True (**T**) or False (**F**). Twelve problems (5 points each) — but there is a penalty for guessing: Credit $= (5 \times \#correct - 2 \times \#wrong)$, minimum credit $= 0$.

Score	
A	
B-1	
B-2	
B-3	
B-4	
B-5	
C-1	
C-2	
C-3	
C-4	
Total	

In problems A-1 through A-4 the matrix A is *similar* to the matrix $C = \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}$.

A-1. T F $A^2 = 3A$.

A-2. T F $\det A = 0$.

A-3. T F $\lambda = 3$ is an eigenvalue of A.

A-4. T F $V = (1, 0)$ is an eigenvector of A.

A-5. T F For a Markov matrix, $\lambda = 1$ is always an eigenvalue.

A-6. T F If the eigenvalues of a matrix are all distinct, then it is similar to a diagonal matrix.

A-7. T F If a matrix is invertible, then it is diagonalizable.

A-8. T F If zero is an eigenvalue of a matrix, then the matrix is *not* invertible.

A-9. T F If V is a given vector in $\mathbb{R}^5$ and W is its orthogonal projection into a two dimensional subspace. Then $\|W\| \le \|V\|$.

A-10. T F If R is an orthogonal matrix, then $\|RX\| = \|X\|$ for *all* vectors X.

A-11. T F Every real symmetric matrix is similar to some diagonal matrix.

A-12. T F If a real matrix A is positive definite, then *all* of its entries must be *positive*.

PART B. Five short-answer questions. 10 points each. Partial credit will rarely be given.

B-1. If λ is an eigenvalue of the $n \times n$ matrix A, show that λ^2 an eigenvalue of A^2.

B-2. Let A be an invertible matrix. If V is a vector with the property that $\langle V, AX \rangle = 0$ for *all* vectors X, show that V must be the zero vector, $V = 0$.

B-3. In $\mathbb{R}^n$, if the vectors X and Y satisfy the Pythagorean relation $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$, show that X and Y are orthogonal.

B-4. Let B be any real matrix, not necessarily square. If its null space is zero, show that $M := B^T B$ is positive definite.

B-5. In the following system of equations, use that the "columns" are orthogonal vectors to solve for z (*only*) [multiple choice]:

$$\begin{aligned} x + y - z - w &= 1 \\ x + y + z + w &= 3 \\ x - y + z - w &= 0 \\ x - y - z + w &= -5 \end{aligned}$$

a) $z = 7$ b) $z = -\frac{3}{2}$ c) $z = -\frac{3}{4}$ d) $z = \frac{7}{4}$ e) $z = -3$ f) $z = \frac{7}{2}$ g) $z = 0$

PART C. Four problems. 20 points each, Please $\boxed{\text{box}}$ your answers where appropriate.

C-1. Let $A := \begin{pmatrix} -2 & c \\ c & -2 \end{pmatrix}$, where c is a constant. To save time, you are given that $V_1 = (1,\, 1)$ and $V_2 = (1,\, -1)$ are both eigenvectors of A.

a). What are the corresponding eigenvalues of A?

b). Consider the system of differential equations $\dfrac{dU}{dt} = AU$ for the vector $U(t)$. Find *all* values of the parameter c so that $\lim_{t\to\infty} U(t) = 0$.

C-2. Say you seek a parabola of the *special form* $y = ax + bx^2$ to pass through the three data points $(-1,\, 2)$, $(0,\, 1)$, $(1,\, 4)$.

a). Write the (over-determined) system of equations you would like to solve ideally.

b). Using the method of least squares, write the normal equations for the coefficients a, b.

c). Explicitly find the coefficients a and b.

C-3. The function $f(x, y, z) = x^4 + y^4 - 4xy + z^2 - 2z - 7$ has critical points at $P_1 = (0, 0, 1)$, $P_2 = (1, 1, 1)$, and $P_3 = (-1 - 1, 1)$. Classify them (*circle* your result).

$P_1 :$ max min saddle $P_2 :$ max min saddle $P_3 :$ max min saddle

C-4. Recall that a real square matrix A is called *anti-symmetric* (or *skew-adjoint)* if $A^T = -A$.
a) If A is any anti-symmetric matrix, show that $\langle X, AX \rangle = 0$ for *all* vectors X.

b). Say a vector $X(t)$ satisfies the differential equation $\dfrac{dX}{dt} = AX$, where A is anti-symmetric. Show that $\|X(t)\| = \text{const.}$. [HINT: Take the derivative of something—and use part a).]