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Math 312
April 23, 1998

Hour Exam 2

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12:00 –1:20

DIRECTIONS: This exam has three parts. Part A has 12 True-False questions (5 points each), Part B has 5 short answer questions (10 points each), and Part C has 4 traditional problems (20 points each). Thus, 190 points total.

To receive full credit your solution must be clear and correct. No fuzzy reasoning. Partial credit will *only* be given for the problems in Part C. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3×5 card with notes.

> NOTE: *To be fair to everyone, those who submit their exam paper late* (after 1:20) *will be "charged" 5 points for every 2 additional minutes.*

PART A. *Circle* True (**T**) or False (**F**). Twelve problems (5 points each) — but there is a penalty for guessing: Credit = $(5 \times \#correct - 2 \times \#wrong)$, minimum credit $= 0$.

In problems A-1 through A-4 the matrix A is *similar* to the matrix $C = \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}$.

A-1. $A^2 = 3A$.
A-2. $\det A = 0$.
A-3. $\lambda = 3$ is an eigenvalue of A.
A-4. $V = (1,\, 0)$ is an eigenvector of A.

A-5. For a Markov matrix, $\lambda = 1$ is always an eigenvalue.
A-6. If the eigenvalues of a matrix are all distinct, then it is similar to a diagonal matrix.
A-7. If a matrix is invertible, then it is diagonalizable.
A-8. If zero is an eigenvalue of a matrix, then the matrix is *not* invertible.
A-9. If V is a given vector in $\mathbb{R}^5$ and W is its orthogonal projection into a two dimensional subspace. Then $\|W\| \le \|V\|$.
A-10. If R is an orthogonal matrix, then $\|RX\| = \|X\|$ for *all* vectors X.
A-11. Every real symmetric matrix is similar to some diagonal matrix.
A-12. If a real matrix A is positive definite, then *all* of its entries must be *positive*.

ANSWERS: All are true *except* A-4, A-7, and A-12.

PART B. Five short-answer questions. 10 points each. Partial credit will rarely be given.

B-1. If λ is an eigenvalue of the $n \times n$ matrix A, show that λ^2 an eigenvalue of A^2.

[*Soln*: If $AV = \lambda V$, then $A^2V = A(AV) = A(\lambda V) = \lambda^2 V$,]

B-2. Let A be an invertible matrix. If V is a vector with the property that $\langle V,\, AX\rangle = 0$ for *all* vectors X, show that V must be the zero vector, $V = 0$.

[*Soln*: Since A is invertible, we can pick X so that $AX = V$. Then $0 = \langle V,\, AX\rangle = \langle V,\, V\rangle = \|V\|^2$ so $V = 0$.]

B-3. In $\mathbb{R}^n$, if the vectors X and Y satisfy the Pythagorean relation $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$, show that X and Y are orthogonal.

[*Soln*: Note that $\|X+Y\|^2 = \langle X+Y,\, X+Y\rangle = \|X\|^2 + 2\langle X,\, Y\rangle + \|Y\|^2$. Comparing with what was given we find that $\langle X,\, Y\rangle = 0$ so X and Y are perpendicular.]

B-4. Let B be any real matrix, not necessarily square. If its null space is zero, show that $M := B^T B$ is positive definite.

[*Soln*: $\langle X,\, MX\rangle = \langle X,\, B^T BX\rangle = \langle BX,\, BX\rangle = \|BX\|^2 \geq 0$. Thus M is at least positive semi-definite.
It is positive definite since if $\langle X,\, MX\rangle = 0$, then $\|BX\| = 0$ so $BX = 0$. But since the null space of B is zero this implies that $X = 0$.]

B-5. In the following system of equations, use that the "columns" are orthogonal vectors to solve for z (*only*) [multiple choice]:

$$\begin{aligned} x+y-z-w &= 1 \\ x+y+z+w &= 3 \\ x-y+z-w &= 0 \\ x-y-z+w &= -5 \end{aligned}$$

a) $z=7$ b) $z=-\frac{3}{2}$ c) $z=-\frac{3}{4}$ d) $z=\frac{7}{4}$ e) $z=-3$ f) $z=\frac{7}{2}$ g) $z=0$

[*Soln*: Let $V_1, \ldots V_4$ be the four column vectors in these equations and let G denote the vector on the right side. Then these equations say that $xV_1 + yV_2 + zV_3 + wV_4 = G$. Since the V_j's are orthogonal, then taking the inner product with V_3 we find that $z\|V_3\|^2 = \langle G,\, V_3\rangle$, so by a computation, $4z = 7$. Thus $z = 7/4$.]

PART C. Four problems. 20 points each, Please box your answers where appropriate.

C-1. Let $A := \begin{pmatrix} -2 & c \\ c & -2 \end{pmatrix}$, where c is a constant. To save time, you are given that $V_1 = (1,\, 1)$ and $V_2 = (1,\, -1)$ are both eigenvectors of A.

a). What are the corresponding eigenvalues of A?

[*Soln*: Computing AV_1 we get $(c-2)V_1$. Thus $\lambda_1 = c-2$. Similarly $\lambda_2 = -2-c$.]

b). Consider the system of differential equations $\dfrac{dU}{dt} = AU$ for the vector $U(t)$. Find *all* values of the parameter c so that $\lim_{t\to\infty} U(t) = 0$.

[*Soln*: By the usual procedures, $U(t) = ae^{\lambda_1 t}V_1 + be^{\lambda_2 t}V_2$, where a and b are constants. To have $\lim_{t\to\infty} U(t) = 0$, we thus need $\lambda_1 < 0$ and $\lambda_2 < 0$. By a computation this says that $-2 < c < 2$.]

C-2. Say you seek a parabola of the *special form* $y = ax + bx^2$ to pass through the three data points $(-1,\ 2)$, $(0,\ 1)$, $(1,\ 4)$.

a). Write the (over-determined) system of equations you would like to solve ideally.

[*Soln*: $\begin{aligned} -a+b&=2\\ 0+0&=1\\ a+b&=4 \end{aligned}$].

b). Using the method of least squares, write the normal equations for the coefficients a, b.

[Soln. Let $A = \begin{pmatrix} -1 & 0\\ 0 & 0\\ 1 & 1\end{pmatrix}$. and $Y = \begin{pmatrix} 2\\ 1\\ 4\end{pmatrix}$. Then the *normal equations* are $A^T AX = A^T Y$. After a brief computationof $A^T A$ and $A^T Y$, the normal equations are $\begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix} a\\ b\end{pmatrix} = \begin{pmatrix} 2\\ 6\end{pmatrix}$.]

c). Explicitly find the coefficients a and b.

[Soln. From part b) $\boxed{a = 1 \text{ and } b = 3}$.]

C-3. The function $f(x, y, z) = x^4 + y^4 - 4xy + z^2 - 2z - 7$ has critical points at $P_1 = (0, 0, 1)$, $P_2 = (1, 1, 1)$, and $P_3 = (-1 - 1, 1)$. Classify them (*circle* your result).

[Soln. Since $f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$, and $f_z = 2z - 2$, then $f_{xx} = 12x^2$ etc. so the second derivative matrix is

$$f'' = \begin{pmatrix} 12x^2 & -4 & 0\\ -4 & 12y^2 & 0\\ 0 & 0 & 2\end{pmatrix}.$$

Thus $f''(P_1) = \begin{pmatrix} 0 & -4 & 0\\ -4 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}$ while $f''(P_2) = f''(P_3) = \begin{pmatrix} 12 & -4 & 0\\ -4 & 12 & 0\\ 0 & 0 & 2\end{pmatrix}$. By the determinant test (or computing the eigenvalues), $f''(P_1)$ is indefinite while $f''(P_2) = f''(P_3)$ are positive definite. Thus P_1 is a saddle point while P_2 and P_3 are local minima.]

C-4. Recall that a real square matrix A is called *anti-symmetric* (or *skew-adjoint)* if $A^T = -A$.

a) If A is any anti-symmetric matrix, show that $\langle X,\ AX\rangle = 0$ for *all* vectors X.

[*Soln*: $\langle X,\ AX\rangle = \langle A^T X,\ X\rangle = -\langle AX,\ X\rangle = -\langle X,\ AX\rangle$. Thus $2\langle X,\ AX\rangle = 0$ so $\langle AX,\ X\rangle = 0$.]

b). Say a vector $X(t)$ satisfies the differential equation $\dfrac{dX}{dt} = AX$, where A is anti-symmetric. Show that $\|X(t)\| = \text{const.}$. [HINT: Take the derivative of something—and use part a).]

[*Soln*: To show that $\|X(t)\|$ is a constant, we show that the derivative of $\|X(t)\|^2$ is zero. But $\frac{d}{dt}\|X(t)\|^2 = \frac{d}{dt}\langle X(t),\ X(t)\rangle = 2\langle X,\ X'\rangle = 2\langle X,\ AX\rangle = 0$ by part a)].